Stochastic Integrals.

In defining the integral

$$I = \int_0^T f(s) dx(s)$$

the Lebesgue theory assumes that $f(\cdot)$ is a bounded continuous function and $x(s)$ is a function of bounded variation so that dx can be thought of as a measure on $[0, T]$. With that we get the bound

$$|I| \leq \sup_{0 \leq s \leq T} |f(s)| Var_{[0,T]} x(\cdot) \tag{1}$$

where

$$Var_{[0,T]} x(\cdot) = \sup_{\mathcal{P}} \sum_{i=0}^{n-1} |x(t_{i+1}) - x(t_i)|$$

where $\mathcal{P} = \{0 = t_0 < t_1 < t_2 \cdots < t_n = T\}$ is an arbitrary partition of $[0, T]$. It is calculated as the limit of the sum

$$I_n = \sum f(x(\tau_i))[x(t_{i+1}) - x(t_i)]$$

where $t_i \leq \tau_i \leq t_{i+1}$ are arbitrary points. The estimate (1) comes from the worst possible case where there are no cancellations and

$$\begin{aligned}
|I| = \lim |I_n| &\leq \sum_{i=0}^{n-1} |f(x(\tau_i))||x(t_{i+1}) - x(t_i)| \\
&\leq \sup_{0 \leq s \leq T} |f(s)| \lim_{n \to \infty} \sum_{i=0}^{n-1} |x(t_{i+1}) - x(t_i)| \\
&= \sup_{0 \leq s \leq T} |f(s)| Var_{[0,T]} x(\cdot)
\end{aligned}$$

However if $x(\cdot)$ is random we can do better. Let $x(\cdot)$ be Brownian motion. we will define the "Wiener Stochastic Integral"

$$I(f) = \int_0^T f(s)\, dx(s)$$

Let us assume that f is a simple function $f = f_i$ on $[t_i \leq t \leq t_{i+1}]$. Then

$$I_(f) = \sum_i f_i [x(t_{i+1}) - x(t_i)]$$

is a Gaussian random variable with mean 0 and variance

$$E[I(f)^2] = \sum_i |f_i|^2 (t_{i+1}) - t_i) = \int_0^T |f(t)|^2 dt$$

If f is a square integrable function on $[0,T]$ it can be approximated by a sequence f_n of simple functions in $L_2[0,T]$. Then

$$\lim_{n,m\to\infty} E[|I(f_n) - I(f_m)|^2] = \lim_{n,m\to\infty} \int_0^T |f_n(t) - f_m(t)|^2 dt = 0$$

Hence

$$I(f) = \lim_{n\to\infty} I(f_n)$$

exists on $L_2(P)$ where the Brownian motion is defined so long as $\int |f(t)|^2 dt < \infty$. $I(f)$ has a Gaussian distribution with mean 0 and variance $\int_0^T |f(t)|^2 dt$. Note that each $I(f)$ is defined only almost surely as a member in $L_2(P)$.

We need to do this because almost surely Brownian motion is NOT of bounded variation. Since Brownian paths are almost surely continuous, if they were of bounded variation, then

$$\lim_{n\to\infty} \sum [x(t_{i+1}) - x(t_i)]^2 \le \sup_i |x(t_{i+1} - x(t_i)| Var_{[0,T]}(x(\cdot)) = 0$$

On the other hand

$$\begin{aligned} E[[\sum [x(t_{i+1}) - x(t_i)]^2 - T]^2 &= E[\sum [[x(t_{i+1}) - x(t_i)]^2 - (t_{i+1} - t_i)]^2] \\ &= \sum E[[x(t_{i+1}) - x(t_i)]^2 - (t_{i+1} - t_i)]^2 \\ &= \sum 2(t_{i+1} - t_i)^2 \end{aligned}$$

and therefore $\sum [x(t_{i+1}) - x(t_i)]^2$ goes to T in L_2 and not to 0.

In Wiener's stochastic integral $f(t)$ is a non-random function. But in Itô's theory, $f(t,\omega)$ can depend on the past history up to time t and not on the future. Consider a partition $0 = t_0 < t_1 < \cdots < t_n = T$. Functions $f_i(\omega)$ that are bounded measurable with respect to $\mathcal{F}_{t_i}$. Note that $x(t)$ is a Brownian motion with respect to $\mathcal{F}_t$. Then if

$$f(t,\omega) = \sum_i f_i(\omega) \mathbf{1}_{[t_i, t_{i+1})}(t)$$

i.e. it equals f_i on $[t_i, t_{i+1}]$ then the natural definition of the integral is

$$S(f,[0,T]) = \int_0^T f(s,\omega) dx(s) = \sum_{i=0}^{n-1} f_i(\omega)[x(t_{i+1}) - x(t_i)]$$

We denote the class of such f's by $\mathbf{F}$. The increment $(x(t_{i+1}) - x(t_i))$ is independent of $\mathcal{F}_{t_i}$ and an easy computation yields

$$E[S(f,[0,T]) = 0 \quad \text{and} \quad E[S(f,[0,T]^2] = E[\int_0^T f(s,\omega)^2 ds]$$

and if we define for $t_k \le t \le t_{k+1}$

$$\xi(t) = S(f,[0,t]) = \int_0^t f(s,\omega)dx(s) = \sum_{i=0}^{k-1} f_i(\omega)[x(t_{i+1}) - x(t_i)] + f_k(\omega)[x(t) - x(t_k)]$$

then $(\xi(t), \mathcal{F}_t)$ is a martingale. It is square integrable and

$$\xi(t)^2 - \int_0^t f(s,\omega)^2 ds$$

is again a $\mathcal{F}_t$ martingale. From Doob's inequality

$$E[[\sup_{0\le t\le T} |\xi(t)|]^2] \le 4E[\int_0^T |f(s,\omega)|^2 ds]$$

In order to define the Itô stochastic integral for f we need to know if we can approximate f by $f_n \in \mathbf{F}$ in the sense that

$$E[\int_0^T |f_n(s,\omega) - f(s,\omega)|^2 ds \to 0$$

A function $f : [0,T]\times\Omega \to R$ is progressively measurable if for every $t \in [0,T]$ the function $f(s,\omega) : [0,t]\times\Omega \to R$ is (jointly) measurable as a map of $[[0,t]\times\Omega, \mathcal{B}[0,t]\times\mathcal{F}_t] \to [R, \mathcal{B}(R)]$ If f is progressively measurable then

$$\int_0^t |f(s,\omega)|^2 ds$$

is again progressively measurable and it makes sense to talk about square integrable progressively measurable functions, i.e. those for which

$$E\left[\int_0^T |f(s,\omega)|^2\, ds\right] < \infty$$

Let us call this class $\mathbf{F}_2$. We define stochastic integrals for such functions by the following steps.

Step 1. We need to show that any $f \in \mathbf{F}_2$ can be approximated by $f_n \in \mathbf{F}$ such that

$$\lim_{n\to\infty} \|f_n - f\|^2_{[0,T]} = E\left[\int_0^T |f_n(s,\omega) - f(s,\omega)|^2\, ds\right] = 0$$

It would then follow that $\xi_n(t,\omega)) = \int_0^t f_n(s,\omega)ds$ satisfies

$$\lim_{n,m\to\infty} E[[\sup_{0\le t\le T} |\xi_n(t) - \xi_m(t)|]^2] = 0$$

There is then the limit ξ of ξ_n that is progressively measurable and almost surely continuous. Moreover $\xi(t)$ would be a martingale as would $[\xi(t)]^2 - \int_0^t [f(s,\omega)]^2 ds$.

Step 2. If $f(t,\omega)$ is uniformly bounded and continuous in t for almost all ω, then $f_n(t,\omega) = f(\frac{[nt]}{n},\omega)$ will work.

Step 3. If $f(t,\omega)$ is bounded and progressively measurable then

$$f_n(t,\omega) = \frac{1}{h}\int_{(t-h)\vee 0}^{t} f(s,\omega)ds$$

is an almost surely continuous, bounded, progressively measurable approximation of f.

Step 4. Finally if $f \in \mathbf{F}_2$, we can truncate and choose $f_n = f$ if $|f| \le n$ and $f_n = 0$ otherwise. Then f_n is clearly progressively measurable, bounded and approximates f. This way for each $f \in \mathbf{F}_2$ we define

$$\xi(t,\omega) = \int_0^t f(s,\omega)dx(s)$$

that satisfies the properties: it is almost surely continuous, and with respect to $\mathcal{F}_t$, both $\xi(t,\omega)$ and $[\xi(t,\omega)]^2 - \int_0^t [f(s,\omega)]^2 ds$ are martingales.

If f is bounded then

$$\exp[\lambda x(t) - \frac{\lambda^2}{2}\int_0^t [f(t,\omega)]^2 ds]$$

is a martingale with respect to $\mathcal{F}_t$. In general (if f is unbounded) it may only be a super-martingale (Fatou's lemma).

Example: Let us calculate $\int_0^t x(s)dx(s)$. One can pretend that $x(s)$ is bounded. Then

$$\begin{aligned}
\int_0^t x(s)dx(s) &= \lim_{\mathcal{P}} \sum x(t_j)(x(t_{j+1}) - x(t_j)) \\
&= \lim_{\mathcal{P}} \frac{1}{2}\sum [(x(t_{j+1}) + x(t_j)(x(t_{j+1}) - x(t_j)] \\
&\quad - \lim_{\mathcal{P}} \frac{1}{2}\sum [(x(t_{j+1}) - x(t_j)(x(t_{j+1}) - x(t_j)] \\
&= \lim_{\mathcal{P}} \frac{1}{2}\sum [(x(t_{j+1})^2 - x(t_j)^2] - \lim_{\mathcal{P}} \frac{1}{2}\sum (x(t_{j+1}) - x(t_j))^2 \\
&= \frac{1}{2}[x(t)^2 - t]
\end{aligned}$$

In other words $dx(t)^2 = 2x(t)dx(t) + dt$. More generally Itô's formula says

$$df(x(t)) = f'(x(t))dx(t) + \frac{1}{2}f''(x(t))dt$$

Sketch of proof.

$$\begin{aligned} f(x(t)) - f(x(s)) &= \sum_j f(x(t_{j+1})) - f(x(t_j)) \\ &\simeq \sum f'(x(t_j))[x(t_{j+1})) - x(t_j)] + \frac{1}{2} \sum f''(x(t_j))[x(t_{j+1})) - x(t_j)]^2 \\ &\quad + \sum O(|x(t_{j+1} - x(t_j)|^2) \end{aligned}$$

The first term goes to $\int_0^t f'(x(s))dx(s)$. The third term goes to 0, because $E[|x(t) - x(s)|^3] \simeq t^{\frac{3}{2}}$. The second term goes to

$$\frac{1}{2} \int_0^t f''(x(s))ds$$