11. Dynamic Programming.

If $x(t) = \int_0^t c(s,\omega)d\beta(s)$ and $|c(s,\omega)| \le C$, then it is not hard to see that

$$E[[x(t)]^2] \le CT$$

But actually if $\phi(x)$ is any convex function

$$E[\phi(x(t)] \le \frac{1}{\sqrt{2\pi Ct}} \int \phi(y) e^{-\frac{y^2}{2Ct}} dy$$

The solution $u(s,x)$ of

$$u_s + \frac{C}{2}_{xx}; u(t,x) = \phi(x)$$

is given by the function

$$u(s,x) = \frac{1}{\sqrt{2\pi C(t-s)}} \int \phi(y) e^{-\frac{(y-x)^2}{2C(t-s)}} dy$$

and $u_{xx} \ge 0$. Therefore

$$du(s,x(s)) - C\,ds = [u_s + \frac{c^2(s,\omega))}{2} u_{xx}(s,x(s)) - C]ds + c(s,\omega)dx(s)$$

and $u(s,x(s))$ is a super-martingale. Therefore

$$E[u(t,x(t)) - Ct] = E[\phi(x(t)) - Ct] \le E[u(0,x(0))] = \frac{1}{\sqrt{2\pi Ct}} \int \phi(y) e^{-\frac{y^2}{2Ct}} dy$$

More generally suppose we have a control u that we can deploy and depending on the control we have an equation

$$dx(s) = d\beta(s) + u(s,\omega)ds; \quad x(0) = x$$

where u can depend on the past history as well. The control has a cost $c(u,x)$ ant the net value is

$$E[f(x(T)) - \int_0^T c(u(s,\omega), x(s,\omega))ds]$$

that we want to optimize. Solve

$$v_t + \frac{1}{2} v_{xx} + h(x, v_x) = 0; v(T,x) = f(x)$$

where

$$h(x,p) = \sup_u [up - c(x,u)]$$

Then for any choice of $c(s,\omega)$,

$$v(t,x(s)) - \int_0^t c(u(s,\omega), x(s,\omega))ds$$

is a super-martingale. Therefore

$$v(0,x) \ge E[f(x(T)) - \int_0^T c(u(s,\omega), x(s,\omega))ds$$