Section 10. Connections with PDE.

We have a progressively measurable stochastic process $x(t,\omega)$ on $(\Omega, \mathcal{F}_t, P)$ such that the paths are continuous with probability 1. We have bounded progressively measurable functions $b(t,\omega)$ and $a(t,\omega)$ with $a(t,\omega) \geq 0$. Moreover

$$y(t) = x(t) - x(0) - \int_0^t b(s,\omega)ds \tag{1}$$

and

$$y^2(t) - \int_0^t a(s,\omega)ds \tag{2}$$

are martingales with respect to $(\Omega, \mathcal{F}_t, P)$. It follows that

$$\exp\left[\theta[x(t) - x(0) - \int_0^t b(s,\omega)ds] - \frac{\theta^2}{2}\int_0^t a(s,\omega)ds]\right]$$

is a martingale for all real θ. We proved it for the special case of $b = 0$ and $a = 1$. If we are in d dimensions $x(t,\omega)$ and $b(t,\omega)$ would be R^d valued and $a = \{a_{i,j}\}$ would be a positive semi-definite matrix. The conclusion would then be

$$\exp\left[\langle\theta, x(t) - x(0)\rangle - \int_0^t \langle\theta, b(s,\omega)\rangle ds] - \frac{1}{2}\int_0^t \langle\theta, a(s,\omega)\theta\rangle ds\right] \tag{3}$$

are martingales with respect to $(\Omega, \mathcal{F}_t, P)$. From this it would then follow that

$$\exp\left[f(t,x(t)) - f(0,x(0)) - \int_0^t [e^{-f(s,x(s))}(\frac{\partial}{\partial t} + L_{s,\omega}e^f)(s,x(s))]ds\right] \tag{4}$$

is again a martingale with respect to $(\Omega, \mathcal{F}_t, P)$ for any smooth f. Replacing f by $\langle\theta, x\rangle + \lambda f(t,x)$ yields more martingales.

$$\exp\left[\lambda[f(t,x(t)) - f(0,x(0))] + \langle\theta, x(t) - x(0)\rangle - \int_0^t [\langle\tilde{\theta}, \tilde{b}(s,\omega)\rangle ds - \frac{1}{2}\int_0^t \langle\tilde{\theta}, \tilde{a}(s,\omega), \tilde{\theta}\rangle]ds\right]$$

Here $\tilde{\theta} = [\lambda, \theta]$, $\tilde{b}(s,\omega) = [f_s + (L_{s,\omega}f)(s,x(s)), b(s,\omega)]$,

$$\tilde{a}(s,\omega) = \begin{pmatrix} \langle a(s,\omega)\nabla f(s,x(s)), \nabla f(s,x(s))\rangle & [a(s,\omega)\nabla f(s,x(s))] \\ [a(s,\omega)\nabla f(s,x(s))]^t & a(s,\omega) \end{pmatrix}$$

and $L_{s,\omega}f$ is the operator

$$(L_{s,\omega}f)(s,x) = \frac{1}{2}\sum_{i,j} a_{i,j}(s,\omega)(D_{x_i}D_{x_j}f)(s,x) + \sum_j b_j(s,\omega)(D_{x_j}f)(s,x)$$

With $x_0(t) = f(t, x(t))$, we define the stochastic integral

$$z(t) = \int_0^t dx_0(s) - \int_0^t \langle \nabla f(s, x(s)), dx(s) \rangle$$

Since $\tilde{a}$ applied to $[1, -\nabla f(s, x(s))]$ is 0, $z(t)$ is of bounded variation and

$$z(t) = z(0) + \int_0^t f_s(s, x(s))ds + \int_0^t (L_{s,\omega} f)(s, x(s)) - \int_0^t \langle b(s, \omega), (\nabla f)(s, x(s)) \rangle ds$$

This yields Itô's formula

$$df(t, x(t)) = f_t(t, x(t))dt + (\nabla f)(t, x(t))dx(t) + \frac{1}{2} \sum_{i,j} a_{i,j}(s, \omega)(D_{x_i} D_{x_j} f)(t, x(t))dt$$

Why does (3) imply (4) ? To see this let us, for simplicity, suppose that $d = 1$ and f does not depend on t. we can replace θ by $i\theta$. This gives us the martingales

$$M_\theta(t) = \exp \left[i\theta[x(t) - x(0) - \int_0^t b(s, \omega)ds] + \frac{\theta^2}{2} \int_0^t a(s, \omega)ds] \right]$$

We can take

$$A(t) = \exp \left[i\theta \int_0^t b(s, \omega)ds] - \frac{\theta^2}{2} \int_0^t a(s, \omega)ds] \right]$$

then the martingale $M(t)A(t) - \int_0^t M(s)dA(s)$ which reduces to

$$f(x(t)) - f(x(0)) - \int_0^t [(L_{s,\omega} f)(s, x(s))]ds$$

with $f(x) = e^{i\theta x}$ is again a martingale. By Fourier integral representation any smooth function is a super position of exponentials $e^{i\theta x}$. The martingale property extends by linearity. Therefore

$$f(x(t)) - f(x(0)) - \int_0^t [(L_{s,\omega} f)(s, x(s))]ds$$

are martingales. Taking e^f instead of f we will get

$$N(t) = e^{f(x(t))} - e^{f(x(0))} - \int_0^t (L_{s,\omega} e^f)(x(s))ds$$

are martingales. Now take

$$A(t) = exp[-f(x(0)) - \int_0^t [(e^{-f} L_{s,\omega} e^f)(x(s))ds]$$

and

$$N(t)A(t) - \int_0^t N(s)dA(s)$$

reduces to what we want. The important thing here is that a continuous process $x(t,\omega)$ with $b(t,\omega)$ and $a(t,\omega)$ representing the conditional infinitesimal mean and covariance in the sense described above is connected very closely to the operator $L_{s,\omega}$. Of course for $L_{s,\omega}$ to be really an operator it is important to have $b(s,\omega) = b(s, x(s,\omega))$ and $a(s,\omega) = a(s, x(s,\omega)$. Then the process is expected to be a Markov process and could have arisen as a solution of a stochastic differential equation

$$dx(t) = \sigma(t, x(t)) \cdot d\beta(t) + b(t, x(t))dt$$

where $\sigma\sigma^* = a$.

There are a few simple formal rules that summarize Itô's formula. Suppose $\beta(t)$ is a Brownian motion then

$$d\beta(t)^2 = dt$$

$\{\beta_i(\cdot)\}$ are independent Brownian Motions

$$d\beta_i(t)d\beta_j(t) = \delta_{i,j}dt$$

and $(dt)^2 = d\beta(t)dt = 0$. Consequently, if

$$dx(t) = a(t,\omega)dt + \sum_i \sigma_i(t,\omega)d\beta_i$$

and

$$dy(t) = b(t,\omega)dt + \sum_i c_i(t,\omega)d\beta_i$$

then

$$dx(t)dy(t) = [\sum_i \sigma_i(t,\omega)c_i(t,\omega)]dt$$

Finally

$$df(x(t)) = (\nabla f)(x(t)) \cdot dx(t) + \frac{1}{2}\sum_{i,j}(D_{x_i}D_{x_j}f)(dx_i(t)dx_j(t))$$

Given $a(s,x)$, and $b(s,x)$, let us define for each s the differential operator

$$L_s = \frac{1}{2}\sum_{i,j} a_{i,j}(s,x)D_{x_i}D_{x_j} + \sum_j b_j(s,x)D_{x_j}$$

Let $u(s,x)$ be a solution of the partial differential equation

$$\frac{\partial u}{\partial s} + (L_s u)(s,x) + g(s,x) = 0; \quad u(T,x) = f(x)$$

Then if $x(t,\omega)$ is any almost surely continuous process satisfying (1), (2) and $P[x(0) = x] = 1$, then

$$u(0,x) = E^P[\int_0^T g(s,x(s))ds + f(x(T))]$$

Proof is elementary.

$$\begin{aligned} du(t,x(t)) &= u_t(t,x(t))dt + (\nabla u)(x(t)) \cdot dx(t) + \frac{1}{2}\sum_{i,j}(D_{x_i}D_{x_j}f)(dx_i(t)dx_j(t)) \\ &= g(t,x(t))dt + \langle \nabla u, \sigma^*(t,x(t))d\beta(t)\rangle \end{aligned}$$

Therefore

$$u(t,x(t)) - u(0,x(0)) + \int_0^t g(s,x(s))$$

is a martingale. Equate expectations at $t = 0$ and $t = T$. There are other relations. If

$$\frac{\partial u}{\partial s} + (L_s u)(s,x) + V(s,x)u(s,x) + g(s,x) = 0; \quad u(T,x) = f(x)$$

then

$$u(0,x) = E^P\left[\int_0^T g(s,x(s))exp[\int_0^s V(\tau,x(\tau))d\tau]ds + exp[\int_0^T V(\tau,x(\tau))d\tau]f(x(T))\right]$$

Ex: Work it out. Enough to show that

$$M(t) = u(t,x(t))A(t) + B(t)$$

is a martingale where

$$A(t) = \exp[\int_0^t V(s,x(s))ds]$$

and

$$B(t) = \int_0^t \exp[\int_0^s V(\tau,x(\tau))d\tau]g(s,x(s))ds$$

Calculate $dM(t)$ and keep only the dt terms.

$$\begin{aligned} dM(t) &= A(t)du(t,x(t)) + u(t,x(t))dA(t) + dB(t) \\ &= A(t)(u_t + L_t u)dt + u(t,x(t))A(t)V(t,x(t)) + A(t)g(t,x(t)) \\ &= A(t)[u_t(t,x(t)) + (L_t u)(t,x(t)) + u(t,x(t))V(t,x(t)) + g(t,x(t))]dt \\ &= 0 \end{aligned}$$

Black and Scholes: If $u(t,x)$ solves

$$u_t + \frac{\sigma^2 x^2}{2}u_{xx} = 0$$

and $x(t)$ is the solution of

$$dx(t) = \sigma x(t) d\beta(t) + b(t, x(t))dt$$

then

$$u(t, x(t)) - u(s, x(s)) = \int_s^t u_x(\tau, x(\tau))dx(\tau)$$

Modify to take care of interest rate r. We need, in prices discounted to current value,

$$e^{-rt}u(t, x(t)) - e^{-rs}u(s, x(s)) = \int_s^t e^{-r\tau}u_x(\tau, x(\tau))[dx(\tau) - r\,x(\tau)d\tau]$$

to keep the hedge. In other words we need

$$d[e^{-rt}u(t, x(t))] = e^{-rt}[u_t - ru + u_x dx + \frac{\sigma^2 x^2}{2} u_{xx} dt] = e^{-rt}[u_x dx - r\,x u_x dt]$$

or a solution of

$$u_t + \frac{\sigma^2 x^2}{2} u_{xx} dt + r\,x u_x - ru = 0$$

with $u(T, x) = f(x)$.

We can solve equations in domains as well. The solution of

$$Lu = \frac{1}{2}\sum_{i,j} a_{i,j}(x) D_{x_i} D_{x_j} u + \sum b_j(x) D_{x_j} u = 0 \quad \text{for} \quad x \in \mathbf{D}$$

with $u = f$ on $\partial\mathbf{D}$, is represented as

$$u(x) = E_x[f(x(\tau)]$$

where τ is the stopping time

$$\tau = \inf[t : x(t) \notin \mathbf{D}]$$

and $E_x[\ ]$ refers to expectation relative to the diffusion process corresponding to L starting from the point $x \in \mathbf{D}$.

There is not that much conceptual difference between the time independent and the time dependent cases. We can always add an extra coordinate x_0 and take $b_0 \equiv 1$ and $a_{0,j} \equiv 0$ for all j. Then we changed

$$u_t + L_t u = u_t + \frac{1}{2}\sum_{i,j} a_{i,j}(t, x) D_{x_i} D_{x_j} u + \sum b_j(t, x) D_{x_j} u$$

to

$$\tilde{L}u = \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(x_0, x) D_{x_i} D_{x_j} u + \sum_{j=1}^{d} b_j(x_0, x) D_{x_j} u + D_{x_0} u$$

The matrix $\tilde{a}$ is now degenerate.

The process starting form L can be defined through PDE. Solve

$$u_s + L_s u = 0 \quad \text{for} \quad s \le t \quad \text{and} \quad u(t,x) = f(x) \tag{5}$$

Represent

$$u(s,x) = \int f(y)p(s,x,t,y)dy$$

Show $p \ge 0$, satisfies Chapman-Kolmogorov equations and is nice enough to be the transition probabilities of a process with continuous paths. This will work if a, b are bounded and Hölder continuous and a is uniformly elliptic.

$$\sum_j |b_j(t,x)| \le C$$

for some $C < \infty$. For some C and $\alpha > 0$,

$$\sum_{i,j} |a_{i,j}(t,x) - a_{i,j}(t,y)| + \sum_j |b_j(t,x) - b_j(t,y)| \le C|x-y|^\alpha,$$

and

$$\sum_{i,j} |a_{i,j}(t,x) - a_{i,j}(s,x)| + \sum_j |b_j(t,x) - b_j(s,x)| \le C|s-t|^\alpha,$$

Finally for some $0 < c \le C < \infty$,

$$c\sum_j \xi_j^2 \le \sum_{i,j} a_{i,j}(t,x)\xi_i\xi_j \le C\sum \xi_j^2.$$

SDE may not work here unless $\alpha = 1$. But any process related to $[a,b]$ by (1) and (2) will be unique and be the same as the one coming from PDE. Because the PDE solution $u(t,x)$ of (5) will still have the property that $u(t,x(t))$ is a martingale with respect to any such process.