9. Diffusion proceses.

A diffusion process is a Markov process with continuous paths with values in some R^d. Given the past history up to time s the conditional distribution at a future time t is given by the transition probability $p(s,x,t,dy)$. The $\{p(s,x,t,dy)\}$ can not be arbitrary. They have to be self consistent, i.e.satisfy the Chapman-Kolmogorov equations

$$\int p(s,x,t,dy)p(t,y,u,A) = p(s,x,u,A)$$

for $s < t < u$. Given such a p and an initial distribution α x at time s, one can construct a Markov process for times $t \geq s$ with

$$\begin{aligned} &P[x(t_0) \in A_0, x(t_1) \in A_1, \ldots, x(t_n) \in A_n] \\ &\qquad = \int_{A_0} \int_{A_1} \cdots \int_{A_n} \alpha(dx) p(s,x,t_1,dy_1) \cdots p(t_{n-1}, y_{n-1}, t_n, dy_n) \end{aligned}$$

for $s = t_0 < t_1 < \cdots < t_n$. One needs some regularity conditions to make sure the paths can be chosen to be continuous.

$$\int \|y - x\|^k p(s,x,t,dy) \leq C|t-s|^{1+\delta}$$

for some $k > 1$ and some $\delta > 0$ is enough (Kolmogorov's Theorem).

We would like to identify them through their infinitesimal means $b(t,x) = \{b_j(t,x)\}$ and infinitesimal covariances $a(t,x) = \{a_{i,j}(t,x)\}$. Roughly speaking we want

$$E[x_j(t+h) - x_j(t)|x(t) = x] = hb_j(t,x) + o(h)$$

and

$$E[(x_i(t+h) - x_i(t))(x_j(t+h) - x_j(t))|x(t) = x] = ha_{i,j}(t,x) + o(h)$$

with

$$P[x(s) \in A] = \alpha(A)$$

We saw that one way was to construct a solution of

$$dx(t) = \sigma(t, x(t))d\beta(t) + b(t, x(t))dt; \quad x(s) = x$$

with σ chosen to satisfy $\sigma(t,x)\sigma(t,x)^* = a(t,x)$. We saw that under a Lipschitz condition on σ the solution, exisits, is unique and is a Markov process with continuous paths. It is the diffusion corresponding to $[a(t,x), b(t,x)]$.

1. What if we had two different square roots, both having unique solutions. Are the two solutions " same"?

2. What if I have two square roots one of them has a unique solution and the other does not,

3. When do I have a Lipschitz square root for a?

If either a is Lipschitz and uniformly elliptic, i.e.

$$c\sum_j \xi_j^2 \le \sum a_{i,j}(t,x)\xi_i\xi_j \le C\sum_j \xi_j^2$$

for some $0 < c \le C < \infty$ and all $\xi \in R^d$ or if $\{a_{i,j}\}$ are twice continuously differentiable with a bound on the second derivatives, the positive semidefinite symmetric square root is Lipschitz.

The basic idea is the matrix version of $(\sqrt{f})' = \frac{f'}{2\sqrt{f}}$ and $|f'| \le \sqrt{2C}\sqrt{f}$ for nonnegative functions f. Here C is a bound on the second derivative. This follows from

$$0 \le f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(\xi) \le f(x) + hf'(x) + \frac{C}{2}h^2$$

for all h. This implies

$$2Cf(x) \ge |f'(x)|^2$$

Suppose σ_1 and σ_2 are two square roots. $\sigma_1\sigma_1^* = \sigma_2\sigma_2^* = a$. Then assuming non-degeneracy, $[\sigma_2^{-1}\sigma_1][\sigma_2^{-1}\sigma_1]^* = I$. In other words $\sigma_1 = \sigma_2 J$ where $J(t,x)$ is orthogonal.

$$\begin{aligned} dx(t) &= b(t,x(t))dt + \sigma_1(t,x(t))d\beta(t) \\ &= b(t,x(t))dt + \sigma_2(t,x(t))J(t,x(t))d\beta(t) \\ &= b(t,x(t))dt + \sigma_2(t,x(t))d\beta'(t) \end{aligned}$$

with β' being a Brownian motion. Using a different square root is the same as using the old square root with a different Brownian motion.

Suppose $x(t,\omega)$ is a stochastic integral

$$x(t) = \int_0^t \sigma(s,\omega)\cdot d\beta(s) + \int_0^t b(s,\omega)ds$$

with $\beta(t,\omega)$ being an n-dimensional Brownian motion with respect to $(\Omega, \mathcal{F}_t, P)$ and σ and b are bounded progressively measurable functions. $b : [0,T]\times\Omega \to R^d$ and $\sigma : [0,T]\times\Omega \to M(d,n)$ the space of $d\times n$ matrices. In other words for $1 \le i \le d$

$$x_i(t) = \sum_{j=1}^n \int_0^t \sigma_{i,j}(s,\omega)d\beta_j(s) + \int_0^t b_i(s,\omega)ds$$

Then one can define stochastic integrals with respect to $x(t)$. We write

$$x(t) = y(t) + A(t)$$

where $\{y_i(t)\}$ are martingales and $A_i(t) = \int_0^t b_i(s,\omega)ds$ are of bounded variation. The integrals

$$z(s) = \int_0^t e(s,\omega)dy(s)$$

are well defined and are again martingales. $e(s,\omega)$ can be $M(k,d)$ valued and $z(t,\omega)$ would be a R^k valued martingale.

$$dz = edx = e[\sigma d\beta + bdt] = e\sigma d\beta + ebdt$$

One verifies such things by checking for simple functions and then passing to the limit. The martingale $z(t)$ would have the property

$$x_i(t)x_i'(t) - \int_0^t [\sigma(s,\omega)\sigma^*(s,\omega)]_{i,i'}ds$$

are martingales. In other words

$$dx \times dx = \sigma\sigma^* dt = a(s,\omega)dt$$

and

$$dz \times dz = eae^* dt = (e\sigma)(e\sigma^*)dt$$

Example. Consider the geometric Brownian Motion defined as the solution of

$$dx(t) = \sigma x(t)d\beta(t) + \mu x(t)dt$$

One could rewrite this as

$$\frac{dx(t)}{x(t)} = \sigma d\beta(t) + \mu dt$$

and expect

$$\log x(t) = \log x(0) + \sigma\beta(t) + \mu t$$

But this would be incorrect because

$$d\log x(t) = \frac{dx(t)}{x(t)} - \frac{1}{2}\frac{dt}{[x(t)]^2}[dx(t)]^2 = \frac{dx(t)}{x(t)} - \frac{\sigma^2}{2}dt = \sigma d\beta(t) + (\mu - \frac{\sigma^2}{2})dt$$

In other words

$$x(t) = \exp[\sigma\beta(t) + (\mu - \frac{\sigma^2}{2})t]$$

Girsanov's formula. Let $x(t)$ be a Brownian motion measure on some $(\Omega, \mathcal{F}_t, P)$. If $b(x)$ is a bounded function then

$$R(t,\omega) = \exp[\int_0^t b(x(s))dx(s) - \frac{1}{2}\int_0^t b^2(x(s))ds]$$

is a Martingale and can be used to define a new measure Q that satisfies

$$\frac{dQ}{dP}|_{\mathcal{F}_t} = R(t,\omega)$$

$x(t)$ will not be Brownian motion with respect to Q. But

$$x(t) - \int_0^t b(x(s))ds = y(t)$$

will be a Brownian motion with respect to Q.

This is not hard to see. With respect to P

$$\exp[[\int_0^t (\theta + b(x(s)))dx(s) - \frac{1}{2}\int_0^t (\theta + b(x(s)))^2 ds]$$

are martingales. In other words

$$\exp[\theta[x(t) - x(0) - \int_0^t b(x(s))ds] - \frac{\theta^2 t}{2}]R(t,\omega)$$

are martingles. This means

$$\exp[\theta[x(t) - x(0) - \int_0^t b(x(s))ds] - \frac{\theta^2 t}{2}]$$

are martingales with respect to Q. So $y(t)$ is Brownian motion under Q. This argument shows that if we have a solution of

$$y(t) = x + \int_0^t b(y(s))ds + x(t)$$

where $x(t)$ is Brownian motion then the distribution of $y(t)$ is uniquely determined as Q with Radon-Nikodym derivative $R(t,\omega)$ with respect to Brownian motion.

Take $d = 1$. Given $a(t,x)$ and $b(t,x)$ we want to define a diffusion process with the property that (1) it has continuous paths, (2) given the history up to time t, the conditional distribution of the future increment $x(t+h) - x(t)$ has mean $hb(t,x(t))$ and variance $a(t,x(t))$. How do we make this precise? We look for a process P on $(C[0,T],\mathcal{F}_t)$ such that

1. $P[x(0) \in A] = \mu(A)$ is given.

2. $y(t) = x(t) - x(0) - \int_0^t b(s,x(s))ds$ is a martingale with respect to $(C[0,T],\mathcal{F}_t,P)$

3. $y^2(t) - \int_0^t a(s,x(s))ds$ is a martingale with respect to $(C[0,T],\mathcal{F}_t,P)$

Does this determine P uniquely? If so what other properties tie P to a,b,μ ?