Assignment 9.

1. Let us consider independent random variables $X_i = \pm 1$ with probability $\frac{1}{2}$ each.

$$S_n = X_1 + X_2 + \cdots + X_n$$

and

$$\xi_n(\frac{j}{n}) = \frac{S_j}{\sqrt{n}}$$

ξ_n is interpolated linearly between $\frac{j}{n}$ and $\frac{j+1}{n}$. Prove the estimate

$$E[|\xi_n(t) - \xi_n(s)|^4] \leq C|t-s|^2$$

for the above random walk approximation to Brownian Motion.

2. $x(t)$ is the price of a stock and evolves according to

$$dx(t) = \sigma\, x(t) d\beta(t) + \mu\, x(t) dt, \quad x(0) = x$$

where μ and σ are constants. Show that

$$\lim_{t\to\infty} \frac{1}{t} E[\log x(t)] = \mu - \frac{\sigma^2}{2}$$

If the current asset $A(t)$ is invested partly (proportion of $\pi(t)$) in stock and partly in cash with an interest rate of r then

$$dA(t) = \pi(t)A(t)[\sigma\, d\beta(t) + \mu dt] + (1-\pi(t))A(t) r dt$$

If $\pi(t)$ is kept constant. i.e. $\pi(t) = \pi$, with $0 \leq \pi \leq 1$, what is the limit

$$\lim_{t\to\infty} \frac{1}{t} E[\log A(t)] = G(\pi)$$

and what is the optimal π that maximizes $G(\pi)$? In principle $\pi = \pi(t,\omega)$ can be progressively measurable. Is keeping it constant the best strategy to maximize $E[\log A(T)]$ for fixed T?