Assignment 4.

1. Consider the region $t \geq 0, |x| \leq 1$. Let $\tau = \inf\{t : |x(t)| = 1\}$. Show that for any staring point $|x| < 1$.

$$P[\tau < \infty | x(0) = x] = 1$$

If $u(t,x)$ is a bounded solution of

$$u_t + \frac{1}{2} u_{xx} = 0, u(t, \pm 1) = 0 \quad \text{for } t > 0$$

then show that $u \equiv 0$. Can you construct a non-trival solution if u is allowed to grow as $t \to \infty$?

Hint: Try a function of the form $f(t)g(x)$.

2. Suppose the region is the wedge $|x| \leq t, t \geq 0$. Can you construct a bounded solution of

$$u_t + \frac{1}{2} u_{xx} = 0, u(t, \pm t) = 0 \quad \text{for } t > 0$$

Hint: Try showing that if $\tau = \inf\{t : |x(t)| = t\}$, then

$$u(t,x) = P[\tau = +\infty | x(t) = x] > 0$$

at least for $x = 0$ and t large. This requires the estimation of

$$\begin{aligned} P[|x(t)| \leq t \ \forall \ t \geq s | x(s) = 0] &\geq 1 - 2P[\sup_{t \geq s} [x(t) - t] \geq 0 | x(s) = 0] \\ &= 1 - 2P[[\sup_{t \geq s} e^{x(t) - t} \geq 1 | x(s) = 0] \\ &\geq 1 - 2P[\sup_{t \geq s} e^{x(t) - \frac{t}{2}} \geq e^{\frac{s}{2}} | x(s) = 0] \end{aligned}$$

Use Doob's inequality.