## 6. Brownian Motion.

A stochastic process can be thought of in one of many equivalent ways. We can begin with an underlying probability space $(\Omega, \Sigma, P)$ and a real valued stochastic process can be defined as a collection of random variables $\{x(t, \omega)\}$ indexed by the parametr set $\mathbf{T}$. This means that for each $t \in \mathbf{T}$, $x(t, \omega)$ is a measurable map of $(\Omega, \Sigma) \to (\mathbf{R}, \mathcal{B}_0)$ where $(\mathbf{R}, \mathcal{B}_0)$ is the real line with the usual Borel $\sigma$-field. The parameter set usually represents time and could be either the integers representing discrete time or could be $[0, T]$, $[0, \infty)$ or $(-\infty, \infty)$ if we are studying processes in continuous time. For each fixed $\omega$ we can view $x(t, \omega)$ as a map of $\mathbf{T} \to \mathbf{R}$ and we would then get a *random function* of $t \in \mathbf{T}$. If we denote by $\mathbf{X}$ the space of functions on $\mathbf{T}$, then a stochastic process becomes a measurable map from a probability space into $\mathbf{X}$. There is a natural $\sigma$-field $\mathcal{B}$ on $\mathbf{X}$ and measurability is to understood in terms of this $\sigma$-field. This natural $\sigma$-field, called the Kolmogorov $\sigma$-field, is defined as the smallest $\sigma$-field such that the projections $\{\pi_t(f) = f(t)\,;\, t \in \mathbf{T}\}$ mapping $\mathbf{X} \to \mathbf{R}$ are measurable. The point of this definition is that a random function $x(\cdot, \omega) : \Omega \to \mathbf{X}$ is measurable if and only if the random variables $x(t, \omega) : \Omega \to \mathbf{R}$ are measurable for each $t \in \mathbf{T}$.

The mapping $x(\cdot, \cdot)$ induces a measure on $(\mathbf{X}, \mathcal{B})$ by the usual definition

$$Q(A) = P\big[\omega : x(\cdot, \omega) \in A\big)$$

for $A \in \mathcal{B}$. Since the underlying probability model $(\Omega, \Sigma, P)$ is irrelevent, it can be replaced by the *canonical* model $(\mathbf{X}, \mathcal{B}, Q)$ with the special choice of $x(t, f) = \pi_t(f) = f(t)$. A stochastic process then can be defined simply as a probability measure $Q$ on $(\mathbf{X}, \mathcal{B})$.

Another point of view is that the only relevent objects are the joint distributions of $\{x(t_1, \omega), x(t_2, \omega), \cdots, x(t_k, \omega)\}$ for every $k$ and every finite subset $F = (t_1, t_2, \cdots, t_k)$ of $\mathbf{T}$. These can be specified as probability measures $\mu_F$ on $\mathbf{R}^k$. These $\{\mu_F\}$ cannot be totally arbitrary. If we allow different permutations of the same set, so that $F$ and $F'$ are permutations of each other then $\mu_F$ and $\mu_{F'}$ should be related by the same permutation. If $F \subset F'$, then we can obtain the joint distribution of $\{x(t, \omega)\,;\, t \in F\}$ by projecting the joint distribution of $\{x(t, \omega)\,;\, t \in F'\}$ from $\mathbf{R}^{k'} \to \mathbf{R}^k$ where $k'$ and $k$ are the cardinalities of $F'$ and $F$ respectively. A stochastic process can then be viewed as a family $\{\mu_F\}$ of distibutions on various finite dimensional spaces that satisfy the consistency conditions. A theorem of Kolmogorov says that this not all that different. Any such consistent family arises from a $Q$ on $(\mathbf{X}, \mathcal{B})$ which is uniquely determine by the family $\{\mu_F\}$.

If $\mathbf{T}$ is countable this is quite satisfactory. $\mathbf{X}$ is the the space of sequences and the $\sigma$-field $\mathcal{B}$ is quite adequate to answer all the questions we may want to ask. The set of bounded sequences, the set of convergent sequences, the set of summable sequences are all measurable subsets of $\mathbf{X}$ and therefore we can answer questions like 'does the sequence converge with probability 1 ?'. etc. However if $\mathbf{T}$ is uncountable like $[0, T]$, then the space of bounded functions, the space of continuous functions etc, are not measurable sets. They do not belong to $\mathcal{B}$. Basically, in probability theory, the rules involve only a countable collection of sets at one time and any information that involves the values of an uncountable number of measurable functions is out of reach. There is an intrinsic reason for this. In probability theory we can change the values of a single random variable on a set of measure 0 and we have not changed anything of consequence. Since we are allowed to mess up each function on a set of measure 0 we have to assume that each function has indeed been messed up on a set of measure 0. If we are dealing with a countable number of functions the 'mess up 'has occured only on the countable union of these invidual sets of measure 0, which by the properties of a measure is again a set of measure 0. On the other hand if we are dealing with an uncountable set of functions, then these sets of measure 0 can possibly gang up on us to produce a set of positive or even full measure. We just can not be sure.

Of course it would be foolish of us to mess things up unnecessarily. If we can clean things up and choose a nice version of our random variables we should do so. But we cannot really do this sensibly unless we decide first what nice means. We however face the risk of being too greedy and it may not be possible to have a version as nice as we seek. But then we can always change our mind.

Very often it is natural to try to find a version that has continuous trajectories. This is equivalent to restricting $\mathbf{X}$ to the space of continuous functions on $[0, T]$ and we are trying to construct a measure $Q$ on $\mathbf{X} = C[0\,,T]$ with the natural $\sigma$-field $\mathcal{B}$. This is not always possible. We want to find some sufficient conditions on the finite dimensional distributions $\{\mu_F\}$ that guarantee that a choice of $Q$ exists on $(\mathbf{X}\,,\mathcal{B})$.

**Theorem 1.** *Assume that for any pair* $(s,t) \in [0\,,T]$ *the bivariate distribution* $\mu_{s,t}$ *satisfies*

$$\int\int |x-y|^{\beta}\mu_{s,t}(dx\,,dy) \le C|t-s|^{1+\alpha}$$

*for some positive constants* $\beta, \alpha$ *and* $C$. *Then there is a unique* $Q$ *on* $(\mathbf{X}\,,\mathcal{B})$ *such that it has* $\{\mu_F\}$ *for its finite dimensional distributions.*

**Proof:** Since we can only deal effectively with a countable number of random variables, we restrict ourselves to valuse at diadic times. Let us for simplicity take $T = 1$. Denote by $\mathbf{T}_n$ time points $t$ of the form $t = \frac{j}{2^n}$ for $0 \le j \le 2^n$. The countable union $\cup_{j=0}^{\infty}\mathbf{T}_j = \mathbf{T}^0$ is a counatble dense subset of $\mathbf{T}$. We will construct a probability measure $Q$ on the space of sequences corresponding to the values of $\{x(t) : t \in \mathbf{T}^0\}$, show that $Q$ is supported on sequences that produce uniformly continuous functions on $\mathbf{T}^0$ and then extend them automatically to $\mathbf{T}$ by continuity and the extension will provide us the natural $Q$ on $C[0\,,1]$. If we start from the set of values on $\mathbf{T}_n$, the $n$-th level of diadics, by linear iterpolation we can construct a version $x_n(t)$ that agrees with the original variables at these diadic points. This way we have a sequence $x_n(t)$ such that $x_n(\cdot) = x_{n+1}(\cdot)$ on $\mathbf{T}^n$. If we can show that for some $\gamma > 0$ and $\delta > 0$,

$$(1) \qquad Q\big[x(\cdot) : \sup_{0\le t\le 1} |x_n(t) - x_{n+1}(t)| \ge 2^{-n\gamma}\big] \le C2^{-n\delta})$$

then we can conclude that

$$Q\big[x(\cdot) : \lim_{n\to\infty} x_n(t) = x_\infty(t) \quad \text{exists uniformly on } [0\,,1]\big] = 1$$

The limit $x_\infty(\cdot)$ will be continuous on $\mathbf{T}$ and will coincide with $x(\cdot)$ on $\mathbf{T}^0$ thereby establishing our result. Proof of (1) depends on a simple observation. The difference $|x_n(\cdot) - x_{n+1}(\cdot)|$

achieves its maximum at the mid point of one of the diadic intervals determined by $\mathbf{T}_n$ and hence

$$\begin{aligned}\sup_{0\le t\le 1}|x_n(t)-x_{n+1}(t)| &\le \sup_{1\le j\le 2^n}|x_n(\frac{2j-1}{2^{n+1}})-x_{n+1}(\frac{2j-1}{2^{n+1}})|\\ &\le \sup_{1\le j\le 2^n}\max\big\{|x(\frac{2j-1}{2^{n+1}})-x(\frac{2j}{2^{n+1}})|,|x(\frac{2j-1}{2^{n+1}})-x(\frac{2j-2}{2^{n+1}})|\big\}\end{aligned}$$

and we can estimate the left hand side of (1) by

$$\begin{aligned}&Q\big[x(\cdot):\sup_{0\le t\le 1}|x_n(t)-x_{n+1}(t)|\ge 2^{-n\gamma}\big]\\ &\le Q\big[\sup_{1\le i\le 2^{n+1}}|x(\frac{i}{2^{n+1}})-x(\frac{i-1}{2^{n+1}})|\ge 2^{-n\gamma}\big]\\ &\le 2^{n+1}\sup_{1\le i\le 2^{n+1}}Q\big[|x(\frac{i}{2^{n+1}})-x(\frac{i-1}{2^{n+1}})|\ge 2^{-n\gamma}\big]\\ &\le 2^{n+1}2^{n\beta\gamma}\sup_{1\le i\le 2^{n+1}}E^Q\big[|x(\frac{i}{2^{n+1}})-x(\frac{i-1}{2^{n+1}})|^\beta\big]\\ &\le C2^{n+1}\,2^{n\beta\gamma}\,2^{-(1+\alpha)(n+1)}\\ &\le C2^{-n\delta}\end{aligned}$$

provided $\delta\le\alpha-\beta\gamma$. For given $\alpha,\beta$ we can pick $\gamma<\frac{\alpha}{\beta}$ and we are done.

An equivalent version of this theorem is the following.

**Theorem 2.** *If $x(t\,,\omega)$ is a stochastic process on $(\Omega\,,\Sigma\,,P)$ satisfying*

$$E^P\big[|x(t)-x(s)|^\beta\big]\le C|t-s|^{1+\alpha}$$

*for some positive constants $\alpha$, $\beta$ and $C$, then if necessary, $x(t,\omega)$ can be modified for each $t$ on a set of measure zero, to obtain an equivalent version that is almost surely continuous.*

As an important application we consider Brownian Motion, which is defined as a stochastic process that has multivariate normal distributions for its finite dimensional distributions. These normal distributions have mean zero and the variance covariance matrix is specified by $Cov(x(s),x(t))=\min(s,t)$. An elementary calculation yields

$$E|x(s)-x(t)|^4=3|t-s|^2$$

so that Theorem (2) is applicable with $\beta=4$, $\alpha=1$ and $C=3$.

To see that some restriction is needed, let us consider the Poisson process defined as a process with independent increments with the distribution of $x(t)-x(s)$ being Poisson with parameter $t-s$ provided $t>s$. In this case since

$$P[x(t)-x(s)\ge 1]=1-\exp[-(t-s)]$$

we have, for every $n \geq 0$,

$$E|x(t) - x(s)|^n \geq 1 - \exp[-|t - s|] \simeq C|t - s|$$

and the conditions for Theorem (2) are never satisfied. It should not be, because after all a Poisson process is a counting process and jumps whenever the event that it is counting occurs and it would indeed be greedy of us to try to put the measure on the space of continuous functions.

**Remark.** The fact that there cannot be a measure on the space of continuous functions whose finite dimensional distributions coincide with those of the Poisson process requires a proof. There is a whole class of nasty examples of measures $\{Q\}$ on the space of continuous functions such that for every $t \in [0\,,1]$

$$Q\Big[\omega : x(t\,,\omega) \text{ is a rational number } \Big] = 1$$

The difference is that the rationals are dense, whereas the integers are not. The proof has to depend on the fact that a continuous function that is not identically equal to some fixed integer must spend a positive amount of time at nonintegral points. Try to make a rigorous proof using Fubini's theorem.

**Brownian Motion as a Markov Process**

Let

$$p(t, x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}}$$

Then

$$\int p(t, x, y)p(s, y, z)dy = p(t + s.x.z)$$

Moreover, because the Brownian motion $x(t)$ is a process with independent increments, i.e. $x(t) - x(s)$ is independent of $\mathcal{F}_s = \sigma\{x(u) : 0 \leq u \leq s\}$

$$P[x(t) - x(s) \in A|\mathcal{F}_s] = \int_A \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{y^2}{2(t-s)}} dy$$

or

$$P[x(t) \in A|\mathcal{F}_s] = \int_A \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(y-x(s))^2}{2(t-s)}} dy$$

establishing the Markov property.

**Strong Markov Property.** If $\{X_n\}$ is a Markov Chain with transition probability $\pi(x, A)$ and $\tau$ is a stopping time then

$$P[X_{\tau+1} \in A|\mathcal{F}_\tau] = \pi(X_\tau, A)$$

i.e the Markov property is valid for stopping times as well. The process after a stopping time $\tau$ behaves exactly like a process starting afresh from $X_\tau$. The proof is quite simple.

Let us show that $E[f(X_{\tau+1}|\mathcal{F}_\tau] = \int f(y)\pi(X_\tau, dy)$. For $A \in \mathcal{F}_\tau$, $A \cup \{\tau = k\} \in \mathcal{F}_k$. Therefore

$$\begin{aligned}
\int_A [\int f(y)\pi(X_\tau, dy)]dP &= \sum_k \int_{A\cap\{\tau=k\}} [\int f(y)\pi(X_\tau, dy)]dP \\
&= \sum_k \int_{A\cap\{\tau=k\}} [\int f(y)\pi(X_k, dy)]dP \\
&= \sum_k \int_{A\cap\{\tau=k\}} f(X_{k+1})\, dP \\
&= \int_A f(X_{k+1})\, dP
\end{aligned}$$

**Exercise:** Show that the process $\{X_{\tau+k} : k \geq 1\}$ conditioned on $\mathcal{F}_\tau$ is the same as the Markov process starting from $X_\tau$.

**Exercise.** In the case of Brownian motion, if $\tau$ is a stopping time that takes only a countable number of values show that the process $x(\tau + t) - x(\tau)$ is again a Brownian motion independent of $\mathcal{F}_\tau$.

**Exercise.** If $\tau$ is a stopping time, then $\tau_n = \frac{[n\tau]+1}{n}$ where $[x]$ is the largest integer not exceeding $x$, is again a stopping time and $\tau_n \downarrow \tau$ as $n \to \infty$.

**Exercise.** Extend the strong Markov property for Brownian motion to any stopping time $\tau$ with $P[\tau < \infty] = 1$.

**Reflection principle.** Let $a > 0$ and

$$\tau = \inf\{t : x(t) \geq a\}$$

Then the set $x(T) \in A$ is the disjoint union of two sets $\{x(T) \in A\} \cap \{\tau \leq T\}$ and $\{x(T) \in A\} \cap \{\tau > T\}$.

$$\int_A \frac{1}{\sqrt{2\pi T}} e^{-\frac{y^2}{2T}}\, dy = \int_A p_1(y) dy + \int_A p_2(dy)$$

By the strong Markov property $p_1(y)$ is symmetric around $y = a$ and $p_2(y)$ is equal to 0 for $y \geq a$. This means

$$p_1(y) = \frac{1}{\sqrt{2\pi T}} e^{-\frac{y^2}{2T}}$$

for $y \geq a$ and for $y \leq a$,

$$p_1(y) = \frac{1}{\sqrt{2\pi T}} e^{-\frac{(2a-y)^2}{2T}}$$

In particular

$$P[\sup_{0\leq t\leq T} x(t) \geq a] = P[\tau \leq T] = \int_{-\infty}^{\infty} p_1(y) dy = 2\int_a^{\infty} \frac{1}{\sqrt{2\pi T}} e^{-\frac{(2a-y)^2}{2T}} dy = 2P[x(T) \geq a]$$

**Brownian Motion As a Martingale.**

If $P$ is the Wiener measure on $(\Omega = C[0,T], \mathcal{B})$ and $\mathcal{B}_t$ is the $\sigma$-field generated by $x(s)$ for $0 \le s \le t$, then $x(t)$ is a martingale with respect to $(\Omega, \mathcal{B}_t, P)$, i.e for each $t > s$ in $[0,T]$

$$E^P\{x(t)|\mathcal{F}_s\} = x(s) \quad \text{a.e.} \quad P$$

and so is $x(t)^2 - t$. In other words

$$E^P\{x(t)^2 - t\,|\mathcal{F}_s\} = x(s)^2 - s \quad \text{a.e.} \quad P$$

The proof is rather straight forward. We write $x(t) = x(s) + Z$ where $Z = x(t) - x(s)$ is a random variable independent of the past history $\mathcal{B}_s$ and is distributed as a Gaussian random variable with mean 0 and variance $t-s$. Therefore $E^P\{Z|\mathcal{B}_s\} = 0$ and $E^P\{Z^2|\mathcal{B}_s\} = t-s$ a.e $P$. Conversely,

**Theorem 3.** ( Levy) *If $P$ is a measure on $(C[0,T], \mathcal{B})$ such that $P[x(0) = 0] = 1$ and the the functions $x(t)$ and $x^2(t) - t$ are martingales with respect to $(C[0,T], \mathcal{B}_t, P)$ then $P$ is the Wiener measure.*

**Proof:** The proof is based on the observation that a Gaussian distribution is determined by two moments. But that the distribution is Gaussian is a consequence of the fact that the paths are almost surely continuous and not part of our assumptions. The actual proof is carried out by establishing that for each real number $l$

$$X_l(t) = \exp\big[lx(t) - \frac{l^2}{2}t\big]$$

is a martingale with respect to $(C[0,T], \mathcal{B}_t, P)$. Once this is established it is elementary to compute

$$E^P\big[\exp\big[l(x(t) - x(s))\big]|\mathcal{B}_s\big] = \exp\big[\frac{l^2}{2}(t-s)\big] \tag{2}$$

which shows that we have a Gaussian Process with independent increments with two matching moments. The proof of (2) is more or less the same as proving the central limit theorem. In order to prove (2) we assume with out loss of generality that $s = 0$ and will show that

$$E^P\big[\exp\big[lx(t) - \frac{l^2}{2}t\big]\big] = 1$$

To this end let us define successively $\tau_{0,\epsilon} = 0$,

$$\tau_{k+1,\epsilon} = \min\big[\inf\big\{s : s \ge \tau_k, |x(s) - x(\tau_{k,\epsilon})| \ge \epsilon\big\}, t\,, \tau_{k,\epsilon} + \epsilon\big]$$

Then each $\tau_{k,\epsilon}$ is a stopping time and eventually $\tau_{k,\epsilon} = t$ by continuity of paths. The continuity of paths also guarantees that $|x(\tau_{k+1,\epsilon}) - x(\tau_{k,\epsilon})| \le \epsilon$. We write

$$x(t) = \sum_{k \ge 0}[x(\tau_{k+1,\epsilon}) - x(\tau_{k,\epsilon})]$$

and

$$t = \sum_{k\geq 0}[\tau_{k+1,\epsilon} - \tau_{k,\epsilon}]$$

To establish (2) we calculate the quantity on the left hand side as

$$\lim_{n\to\infty} E^P\big[\exp\big[\sum_{0\leq k\leq n}\big[l[x(\tau_{k+1,\epsilon}) - x(\tau_{k,\epsilon})] - \frac{l^2}{2}[\tau_{k+1,\epsilon} - \tau_{k,\epsilon}]\big]\big]\big]$$

and show that it equals 1. Let us cosider the $\sigma$-field $\mathcal{F}_k = \mathcal{B}_{\tau_{k,\epsilon}}$ and the quantity

$$q_k(\omega) = E^P\big[\exp\big[l[x(\tau_{k+1,\epsilon}) - x(\tau_{k,\epsilon})] - \frac{l^2}{2}[\tau_{k+1,\epsilon} - \tau_{k,\epsilon}]\big]\Big|\mathcal{F}_k\big]$$

Clearly, if we use Taylor expansion and the fact that $x(t)$ as well as $x(t)^2 - t$ are martingales

$$\begin{aligned}|q_k(\omega) - 1| &\leq CE^P\big[\big[l^3|x(\tau_{k+1,\epsilon}) - x(\tau_{k,\epsilon})|^3 + l^2|\tau_{k+1,\epsilon} - \tau_{k,\epsilon}|^2\big]\Big|\mathcal{F}_k\big] \\ &\leq C_l\,\epsilon\, E^P\big[\big[|x(\tau_{k+1,\epsilon}) - x(\tau_{k,\epsilon})|^2 + |\tau_{k+1,\epsilon} - \tau_{k,\epsilon}|\big]\big|\mathcal{F}_k\big] \\ &= 2C_l\,\epsilon\, E^P\big[|\tau_{k+1,\epsilon} - \tau_{k,\epsilon}|\big|\mathcal{F}_k\big]\end{aligned}$$

In particular for some constant $C$ depending on $l$

$$q_k(\omega) \leq E^P\big[\exp\big[C\,\epsilon\,[\tau_{k+1,\epsilon} - \tau_{k,\epsilon}]\big]\big|\mathcal{F}_k\big]$$

and by induction for every $n$

$$\begin{aligned}&E^P\big[\exp\big[\sum_{0\leq k\leq n}\big[l[x(\tau_{k+1,\epsilon}) - x(\tau_{k,\epsilon})] - \frac{l^2}{2}[\tau_{k+1,\epsilon} - \tau_{k,\epsilon}]\big]\big]\big] \\ &\leq \exp[C\,\epsilon\, t\,]\end{aligned}$$

Since $\epsilon > 0$ is arbitrary we prove one half of (2). Notice that in any case $\sup_\omega |q_k(\omega) - 1| \leq C_l\epsilon^2$. Hence we have the lower bound

$$q_k(\omega) \geq E^P\big[\exp\big[-C\,\epsilon\,[\tau_{k+1,\epsilon} - \tau_{k\ \epsilon}]\big]\Big|\mathcal{F}_k\big]$$

which can be used to prove the other half. The upper bound provides the necessary uniform integrability. This completes the proof of the theorem.

**Exercise:** Why does the Theorem fail for the process $x(t) = N(t) - t$ where $N(t)$ is the standard Poisson Process with rate 1?

**Remark:** One can use the Martingale inequality in order to estimate the probability $P\{\sup_{0\leq s\leq t}|x(s)| \geq \ell\}$. For $l > 0$, by Doob's inequality

$$P\big[\sup_{0\leq s\leq t}\exp\big[lx(s) - \frac{l^2}{2}s\big] \geq A\big] \leq \frac{1}{A}$$

and

$$\begin{aligned}
P\Big[\sup_{0\le s\le t} x(s)\ge \ell\Big] &\le P\Big[\sup_{0\le s\le t}[x(s)-\frac{ls}{2}]\ge \ell-\frac{lt}{2}\Big]\\
&= P\Big[\sup_{0\le s\le t}[lx(s)-\frac{l^2 s}{2}]\ge l\ell-\frac{l^2 t}{2}\Big]\\
&\le \exp[-l\ell+\frac{l^2 t}{2}]
\end{aligned}$$

Optimizing over $l > 0$, we obtain

$$P\Big[\sup_{0\le s\le t} x(s)\ge \ell\Big]\le \exp[-\frac{\ell^2}{2t}]$$

and by symmetry

$$P\Big[\sup_{0\le s\le t} |x(s)|\ge \ell\Big]\le 2\exp[-\frac{\ell^2}{2t}]$$

The estimate is not too bad because by reflection principle

$$P\Big[\sup_{0\le s\le t} x(s)\ge \ell\Big]=2\,P\Big[x(t)\ge \ell\Big]=\sqrt{\frac{2}{\pi\, t}}\int_\ell^\infty \exp[-\frac{x^2}{2\,t}]\,dx$$