Martingales II

Super and sub-martingales. In the definition of martingale we demanded that for every n,

$$E[X_n|\mathcal{F}_{n-1}] = X_{n-1}$$

Instead if for every n

$$E[X_n|\mathcal{F}_{n-1}] \geq X_{n-1}$$

then $\{X_n\}$ is called a sub-martingale and if for every n,

$$E[X_n|\mathcal{F}_{n-1}] \leq X_{n-1}$$

it is called a super-martingale. An important result is Jensen's inequality.

Theorem. If X_n is a martingale and if $\phi(x)$ is a convex function of x then $\phi(X_n) = Y_n$ is a sub-martingale, provided $\phi(X_n)$ is integrable.

Proof: By duality any convex function $\phi(x)$ has a representation

$$\phi(x) = \sup_{\ell}[\ell\, x - \psi(\ell)]$$

for some ψ, (which is convex as well). By linearity of conditional expectation, for every ℓ,

$$E[\ell X_n - \psi(\ell)|\mathcal{F}_{n-1}] = \ell X_{n-1} - \psi(\ell)$$

By monotonicity of condition expectation (a consequence of non-negativity and linearity)

$$E[\phi(X_n)|\mathcal{F}_{n-1}] \geq E[\ell X_n - \psi(\ell)|\mathcal{F}_{n-1}] = \ell X_{n-1} - \psi(\ell)$$

for every ℓ and hence

$$E[\phi(X_n)|\mathcal{F}_{n-1}] \geq \sup_{\ell} E[\ell X_n - \psi(\ell)|\mathcal{F}_{n-1}] = \phi(X_{n-1})$$

For those worried about sets of measure 0, we can limit ourselves to a countable set of values of ℓ.

In particular if X_n is a martingale then for $\alpha \geq 1$, $|X_n|^\alpha$ is a sub-martingale. This observation provides an important inequality.

Theorem. (Doob's Inequality.) Let $\{X_n\}$ be a martingale. Let $\xi_n = \sup_{0 \leq j \leq n} |X_j|$. Then

$$P[\xi_n \geq \ell] \leq \frac{1}{\ell} \int_{|\xi_n| \geq \ell} |X_n| dP$$

Proof: Let us write $E_n = \{\omega : \xi_n \geq n\}$ as the disjoint union of $\{F_j : 0 \leq j \leq n\}$ where

$$F_j = \{\omega : |X_i| < \ell \text{ for } 0 \leq j \leq j-1, |X_j| \geq \ell\}$$

Then

$$P(F_j) \le \frac{1}{\ell}\int_{F_j} |X_j| dP \le \int_{F_j} |X_n| dp$$

because $|X_n|$ is a sub-martingale and $F_j \in \mathcal{F}_j$. Summing ove $0 \le j \le n$ we get

$$P(E_n) \le \int_{E_n} |X_n| dP$$

Lemma: Let Y and ξ be two non-negative random variables such that

$$P[\xi \ge \ell] \le \frac{1}{\ell}\int_{\xi \ge \ell} Y\, dP$$

for $\ell \ge 0$. Then for $p > 1$

$$\int \xi^p\, dP \le \left(\frac{p}{p-1}\right)^p \int Y^p\, dP$$

Proof: We can write

$$\begin{aligned}
\int \xi^p\, dP &= -\int \ell^p dP[\xi \ge \ell] = p\int P[\xi \ge \ell]\ell^{p-1} d\ell \\
&\le p\int\int_{\xi \ge \ell} \ell^{p-2} Y\, dP d\ell = \frac{p}{p-1}\int \xi^{p-1}\, Y\, dP \\
&\le \frac{p}{p-1}\left[\int \xi^p\, dP\right]^{\frac{p-1}{p}} \left[\int Y^p\, dP\right]^{\frac{1}{p}}
\end{aligned}$$

leading to

$$\left[\int \xi^p\, dP\right]^{\frac{1}{p}} \le \frac{p}{p-1}\left[\int Y^p\, dP\right]^{\frac{1}{p}}$$

This proof assumes that $\int \xi^p\, dP < \infty$. If we only assume that $\int Y^p\, dP$ is finite, then we can truncate ξ by $\xi_a = \xi \wedge a$. Then it is easy to verify that ξ_a satisfies the assumptions of the lemma and we get the inequality

$$\left[\int \xi_a^p\, dP\right]^{\frac{1}{p}} \le \frac{p}{p-1}\left[\int Y^p\, dP\right]^{\frac{1}{p}}$$

We let $a \to \infty$ to prove the lemma. A consequence of the lemma is

Theorem. Let $\{X_n\}$ be a martingale. Then for $p > 1$, if $|X_n|$ is in L_p,

$$\int [\sup_{0 \le j \le n} |X_n|]^p\, dP \le \left(\frac{p}{p-1}\right)^p \int |X_n|^p\, dP$$

Doob decomposition. If X_n is any sequence of integrable random variables and X_n is measurable with respect to an increasing family of sub-σ-fields $\mathcal{F}_n \subset \mathcal{F}$ we can write

$$X_n = Y_n + A_n$$

where Y_n is a martingale with respect to $\mathcal{F}_n$ and A_n is $\mathcal{F}_{n-1}$ measurable. Such a decomposition is unique. To see that the decomposition exists define

$$a_n(\omega) = E[X_{n+1} - X_n(\omega)|\mathcal{F}_n]$$
$$A_n = \sum_{j=0}^{n-1} a_j(\omega)$$

Then $X_n = Y_n + A_n$ with$Y_n = (X_n - A_n)$. Clearly A_n is $\mathcal{F}_{n-1}$ measurable and

$$\begin{aligned} E[Y_n|\mathcal{F}_{n-1}] &= E[X_n - A_n|\mathcal{F}_{n-1} = X_{n-1} - A_n + E[X_n - X_{n-1}|\mathcal{F}_{n-1}] \\ &= X_{n-1} - A_n + a_n = X_{n-1} - A_{n-1} = Y_{n-1}. \end{aligned}$$

proving that Y_n is a martingale. As for uniqueness if $X_n = Y_n + A_n$ and Y_n is a martingale, then

$$\begin{aligned} 0 &= E[Y_n - Y_{n-1}|\mathcal{F}_{n-1}] = E[X_n - X_{n-1} - A_n + A_{n-1}|\mathcal{F}_{n-1}] \\ &= E[X_n - X_{n-1}|\mathcal{F}_{n-1}] - A_n + A_{n-1} \end{aligned}$$

establishing that $a_n = A_n - A_{n-1} = E[X_n - X_{n-1}|\mathcal{F}_{n-1}]$. X_n is a sub-martingale if A_n is increasing or $a_n \geq 0$. Similarly X_n is a super-martingale if A_n is decreasing or $a_n \leq 0$.

Infinite martingale sequenecs. One way of generating an infinite martingale sequence X_n is to start with $X(\omega)$ which is integrable on $(\Omega, \mathcal{F}, P)$ and define $X_n = E[X|\mathcal{F}_n]$. It is easy to check from the properties of conditional expectation that $E[X_n|\mathcal{F}_{n-1}] = X_{n-1}$. Assuming that the σ-field $\mathcal{F}$ is generated by $\mathcal{F}_n$, It is a not a difficult result to prove that $X_n \to X$ with probability 1 and in $L_1(P)$. Moreover if for any $1 < p < \infty$ if $X \in L_p$, $X_n \to X$ in $L_p(P)$. The proofs are straight forward and we will give a sketch. Note that from standard measure theory, if $\mathcal{F}$ is generated by $\cup_n \mathcal{F}_n$, then $W = \cup_n L_1(\omega, \mathcal{F}_n, P)$ is dense in $L_1(\Omega, \mathcal{F}, P)$. If $X \in W$ then $X_n = X$ for large n and the convergence is trivial. Then by standard approximation, if $X \in L_1(P)$, approximating it by $Y \in W$, since $Y_n = E[Y|\mathcal{F}_n] \to Y$ in L_1

$$\limsup_{n\to\infty} \|X_n - X\|_1 \leq \|X - Y\|_1 + \limsup_{n\to\infty} \|X_n - Y_n\|_1 \leq 2\|X - Y\|_1$$

Since we can choose Y so that $\|X - Y\|_1$ is small we are done. The same argument works to prove almost sure convergence as well. We note that from Doob's inequality

$$P[\sup_{0\leq j\leq n} |X_j| \geq \ell] \leq \frac{1}{\ell} E[|X_n|] \leq \frac{1}{\ell} E[|X|]$$

Hence $P[\sup_{0\leq j<\infty} |X_j| < \infty] = 1$ and we need only to prove, that for any $\epsilon > 0$,

$$P[\limsup_{n\to\infty} X_n - \liminf_{n\to\infty} X_n \geq \epsilon] = 0$$

Since for $Y \in W$,

$$P[\limsup_{n\to\infty} Y_n - \liminf_{n\to\infty} Y_n \geq \epsilon] = 0$$

it is enough to estimate

$$P[\sup_{1\leq n<\infty} |X_n - Y_n| \geq \epsilon] \leq \frac{1}{\epsilon} E[|X - Y|]$$

which can now be made arbitrarily small by the choice of $Y \in W$.

Given a martinagle sequence X_n does it arise from some X in $L_p(P)$. A necessary condition is that $\|X_n\|_p$ must be uniformly bounded. If it is and $p > 1$, it is also weakly compact in L_p, and a weak limit will produce the needed X. If $p = 1$, uniform integrability is required to establish weak compactness. Otherwise $X_n \to X$ almost surely but not in $L_1(P)$ and X_n does not arise from X by conditional expectation.

Examples and Problems.

1. If λ, μ are two probability measures such that $\lambda << \mu$ on each $\mathcal{F}_n$ but not on $\mathcal{F}$ generated by $\cup_n \mathcal{F}_n$, the radon-Nikodym derivative

$$X_n(\omega) = \frac{d\lambda}{d\mu}\bigg|_{\mathcal{F}_n}$$

is a martingale but cannot come from any X because if it did we would have $X = \frac{d\lambda}{d\mu}\big|_{\mathcal{F}}$ and $\lambda << \mu$ on $\mathcal{F}$.

2. One can take Ω to be the unit interval, $\mathcal{F}$ to be the Borel σ-field and $\mathcal{F}_n$ to be the partition $[\frac{j-1}{2^n}, \frac{j}{2^n}]$ of $[0, 1]$. If μ is Lebesgue measure, the any λ is absoulutely continuous with respect to μ and

$$\frac{d\lambda}{d\mu}\bigg|_{\mathcal{F}_n} = 2^n \lambda([\frac{j-1}{2^n}, \frac{j}{2^n}]) \text{ for } x \in [\frac{j-1}{2^n}, \frac{j}{2^n}]$$

λ can be singular with respect to Lebesgue measure.

3. Let Ω be the space of real sequences $(x_1, \ldots, x_n, \ldots)$ with the product σ-field. P is the product measure of standard normal distributions with mean 0 variance 1. In other words $\{x_i\}$ are i.i.d standard normal variables under P. Show that

$$X_n(\omega) = \exp[\lambda\,(x_1 + \cdots + x_n) - \frac{\lambda^2}{2} n]$$

is a martinagle with respect to $(\Omega, \mathcal{F}_n, P$ where $\mathcal{F}_n$ is the σ-field corresponding to the first n coordinates $x_1, \ldots, x_n$. Is it uniformly integrable? If not is there a Q on $\mathcal{F}$ such that $X_n = \frac{dQ}{dP}|_{\mathcal{F}_n}$. If so, can you describe Q?

4. Go back to example **2** and define $\{x_j\}$ as the entries in the binary expansion of $x \in [0,1]$. What is the joint distribution of $\{x_j\}$ under the Lebesgue measure μ? Can you determine $c(\lambda)$ such that

$$X_n(\omega) = \exp[\lambda\,(x_1 + \ldots + x_n) - c(\lambda)\,n]$$

is a martingale? What should be the measure λ on $[0,1]$ be so that X_n is the Radon-Nikodym derivative $\frac{d\lambda}{d\mu}$ on $\mathcal{F}_n$?

5. Let $X_i = \pm 1$ with probability $\frac{1}{2}$, and be mutually independent. $S_n = x + X_1 + X_2 + \cdots + X_n$ is the standard random walk starting from x. Given $0 < x < N$, what is the probability that S_n reaches 0 before reaching N? Use the stopping time $\tau = \inf\{n : S_n = 0 \text{ or } N\}$, and the martingale property of S_n.

6. S_n^2 is a sub-martingale. What is its Doob decomposition? Can you use it to calculate $E[\tau] = m(x)$?

7. In examples **5** and **6**, while working with τ, which is not a bounded stopping time how can you justify your calculations?