Chapter 1

Poisson Processes

1.1 The Basic Poisson Process

The Poisson Process is basically a counting processs. A Poisson Process on the interval $[0\,,\infty)$ counts the number of times some primitive event has occurred during the time interval $[0\,,t]$. The following assumptions are made about the 'Process' $N(t)$.

(i). The distribution of $N(t+h)-N(t)$ is the same for each $h>0$, i.e. is independent of t.

(ii). The random variables $N(t_j')-N(t_j)$ are mutually independent if the intervals $[t_j\,,t_j']$ are nonoverlapping.

(iii). $N(0)=0$, $N(t)$ is integer valued, right continuous and nondecreasing in t, with Probability 1.

(iv). $P\,[N(t+h)-N(t)\geq 2\,]=P\,[N(h)\geq 2\,]=\,o(h)$ as $h\to 0$.

Theorem 1.1. *Under the above assumptions, the process $N(\cdot)$ has the following additional properties.*

(1). With probability 1, *$N(t)$ is a step function that increases in steps of size* 1.

(2). There exists a number $\lambda\geq 0$ such that the distribution of $N(t+s)-N(s)$ has a Poisson distribution with parameter $\lambda\,t$.

(3). The gaps $\tau_1, \tau_2, \cdots$ between successive jumps are independent identically distributed random variables with the exponential distribution

$$P\{\tau_j \geq x\} = \begin{cases} \exp[-\lambda\, x] & \text{for } x \geq 0 \\ 1 & \text{for } x \leq 0 \end{cases} \tag{1.1}$$

Proof. Let us divide the interval $[0\,,T]$ into n equal parts and compute the expected number of intervals with $N(\frac{(k+1)T}{n}) - N(\frac{kT}{n}) \geq 2$. This expected value is equal to

$$nP\big[N(\frac{T}{n}) \geq 2\,\big] = n\,.\,o(\frac{1}{n}) = o(1) \quad \text{as} \quad n \to \infty$$

there by proving property (1).

Because of right continuity we have

$$P[N(t) \geq 1] \to 0 \quad \text{as} \quad t \to 0$$

proving that the distribution of $N(t)$ is infinitesimal as $t \to 0$. By the independence of the increments over disjoint intervals, $N(t)$ is approximately the sum of $[nt]$ independent copies of $N(\frac{1}{n})$.

$$\begin{aligned} E\big\{\exp[-\sigma N(t)]\big\} &= \lim_{n\to\infty} E\big\{\exp[-\sigma N(\frac{[nt]}{n})]\big\} \\ &= \lim_{n\to\infty} \big[E\big\{\exp[-\sigma N(\frac{1}{n})]\big\}\big]^{[nt]} &(1.2) \\ &= \exp[-tg(\sigma)] &(1.3) \end{aligned}$$

where

$$\begin{aligned} g(\sigma) &= \lim_{n\to\infty} n\Big[1 - \big[E\big\{\exp[-\sigma N(\frac{1}{n})]\big\}\big]\Big] \\ &= \lim_{n\to\infty} n\big[E\big\{1 - \exp[-\sigma N(\frac{1}{n})]\big\}\big] \\ &= \lim_{n\to\infty} n\big[(1-e^{-\sigma})P[N(\frac{1}{n}) = 1] + o(\frac{1}{n})\big] \\ &= \lambda(1-e^{-\sigma}) &(1.4) \end{aligned}$$

where

$$\lambda = \lim_{n\to\infty} n\, P[N(\frac{1}{n}) = 1]. \tag{1.5}$$

The limit in (1.5) must necessarily exist because the limit in (1.3) clearly exists. The positivity of $E\big\{\exp[-\sigma N(t)]\big\}$ and the identity (1.4) guarantees the finiteness of the limit in (1.5). The formulas (1.4) and (1.5) together identify the distribution of $N(t)$ as the Poisson distribution with parameter λt, thereby proving property (2).

Finally, we turn to the proof of prpoerty (3). First let us prove that τ_1 has the right distribution.

$$P[\tau_1 > x] = P[N(x) = 0\,] = e^{-\lambda x}$$

because of the Poisson distribution. We will prove τ_1 is a regenerative time in the sense that $N(\tau_1+t)-N(\tau_1) = N(\tau_1+t)-1$ is again a Poisson Process independent of τ_1. This will prove that τ_2 will have the same distribution as τ_1 and will be independent of it. Repeating the step and induction on n will complete the proof. Let τ be a stopping time that takes a countable set of values $\{v_j\}$. Since the set $\tau = v_j$ is measurable with respect to $\sigma\{N(t) : t \leq v_j\}$, the process $N(t+v_j) - N(v_j)$ is independent of the set $\tau = v_j$, and is conditionally again a Poisson Process. Therefore for such a τ, $N(\tau+t)-N(\tau)$ is again a Poisson process independent of τ. Finally, τ_1 is a stopping time and for any k, $\tau^{(k)} = \frac{[k\tau_1]+1}{k}$ is a stopping time that takes only a countable number of values. Therefore $N(\tau^{(k)}+t) - N(\tau^{(k)})$ is a Poisson Process with parameter λ that is independent of $\tau^{(k)}$. We let $k \to \infty$. Because $\tau^{(k)} \geq \tau_1$, and the process is right continuous we are done. $\square$

Remark 1.1. What we have proved is that if P is a measure on the space Ω of nondecreasing right continuous functions on $[0,\infty)$ satisfying the properties listed in the assumptions of Theorem 1.1, then $P = P_\lambda$ is determined by a single parameter λ and the additional properties (1)-(3) of Theorem 1.1 will be valid for it. The process P_λ is referred to as the Poisson process with parameter or 'rate' λ.

Exercise 1.1. Alternate proof of property (3) of Theorem 1.1. Let $\sigma \leq 0$ be arbitrary. The processes

$$M_\sigma(t) = \exp[\sigma N(t) - \lambda t(e^\sigma - 1)]$$

are martingales with repect to $(\Omega\,, \mathcal{F}_t\,, P)$ and Doob's stopping theorem provides the relation

$$E\{M_\sigma(\tau+t)|\mathcal{F}_\tau\} = M_\sigma(\tau) \ \text{ a.e.}$$

Turn it into a proof that τ_1 is regenerative.

Exercise 1.2. Verify that for any Poisson process with parameter λ

$$N(t) - \lambda \quad \text{and} \qquad [N(t) - \lambda t]^2 - \lambda t$$

are martingales with respect to $(\Omega\,, \mathcal{F}_t\,, P)$, where

$$\mathcal{F}_t = \sigma\{N(s) : 0 \le s \le t\}$$

Exercise 1.3. The distribution of $\tau_1 + \cdots + \tau_{k+1}$ is a Gamma distribution with density $\frac{\lambda^k}{k!} e^{-\lambda x} x^{k-1}$. Why does it look like the Poissson probability for k jumps?

1.2 Compound Poisson Processes.

Suppose that $X_1, X_2 \cdots, X_n \cdots$ is a sequence of independent identically distributed random variables with a common distribution α having partial sums $S_n = X_1 + X_2 + \cdots + X_n$. We define a proces $X(t)$ by

$$X(t) = S_{N(t)}.$$

Each time the Poisson Process $N(t)$ jumps by 1, the process $X(t)$ jumps by a random amount which has distribution α. The jumps at different instances are independent. Such a process $X(t)$ inherits the independent increments property from $N(t)$. For any collection of nonoverlapping intervals $[t_j\,, t'_j]$ the increments $X(t'_j) - X(t_j)$ are independent random variables. The distribution of any increment $X(t+s) - X(s)$ is that of $X_{N(t)}$, and is calculated to be the distribution of S_n where n is random and has a Poisson distribution with parameter λt.

$$E\{\exp[i\, y\, X(t)]\} = \sum_j e^{-\lambda t} \frac{(\lambda t)^j}{j!} [\hat{\alpha}(y)]^j = e^{-\lambda t} e^{\lambda t\, \hat{\alpha}(y)} = e^{\lambda t[\hat{\alpha}(y) - 1]}$$
$$= \exp[\lambda t \int (e^{i\, y\, x} - 1)\, d\alpha(x)]$$

Inother words $X(t)$ has an infinitely divisible distribution with a Levy measure given by $\lambda\, t\, \alpha$. If we denote by $M = \lambda \alpha$ then

$$E\{\exp[i\, y\, X(t)]\} = \exp[t \int (e^{i\, y\, x} - 1)\, dM(x)]$$

Exercise: Assuming that α has two moments, show that are constants A and B such that $X(t)-At$ and $[X(t)-At]^2-Bt$ are martingales. Compute A and B in terms of λ and α. $[A=\int x\,dM=\lambda\int x\,d\alpha\,;\ B=\int x^2\,dM=\lambda\int\ x^2\,d\alpha\,]$

1.3 Infinite number of small jumps.

Where as a Poisson process cannot have an infinite number of jumps in a finite interval, once we start considering compound Poisson processes we can in principle sum an infinite number of small jumps so that we still have a finite answer. For example suppose $X_k(t)$ is a compound Poisson Process that corresponds to $\lambda_k\alpha_k=M_k$.

$$E\{X_k(t)\}=t\int x\,dM_k(x)$$

and

$$Var[X_k(t)]=t\int x^2\,dM_k(x)$$

Let us take $X_k(t)$ to be mutually independent processes with independent increments and try to sum up

$$X(t)=\sum_k X_k(t)$$

for each t. If the sum exists then it is a process with independent increments. In fact we can do a little better. We may center these processes with suitable constants $\{a_k\}$, and write

$$X(t)=\sum_k[X_k(t)-a_kt]$$

to help in convergence. We shall assume that

$$\sum_k\int\frac{x^2}{1+x^2}dM_k(x)<\infty$$

which amounts to two conditions

$$\sum_k M_k[|x|\geq 1]<\infty$$

and

$$\sum_k \int_{|x|\le 1} x^2 dM_k(x) < \infty$$

We can always decompose any M as $M_1 + M_2$ and this will result in a decomposition of the process $X(t) = X_1(t) + X_2(t)$, two mutually independent processes with independent increments corresponding to M_1 and M_2 respectively. It is better for us to decompose each M_k as $M_k^{(1)} + M_k^{(2)}$ corresponding to jumps of size $|x| \le 1$ and $|x| > 1$.

$$M = \sum_k M_k^{(2)}$$

sums to a finite measure and offers no difficulty. The process

$$X^{(2)}(t) = \sum_k X_k^{(2)}(t)$$

exists very nicely because

$$P\{X_k^{(2)}(t) \ne 0\} \le 1 - e^{-tM_k^{(2)}[\,|x|>1\,]} \le tM_k^{(2)}[|x| > 1]$$

and

$$\sum_k P\{X_k^{(2)}(t) \ne 0\} < \infty$$

Borel-Cantelli does it! As for a discussion of $\sum_k X_k^{(1)}(t)$, it is now clear that we can assume that $M_k^{(1)}[\,|x| > 1\,] = 0$ for every k. If we take $a_k = E\{X_k(1)\} = \int x\, dM_k^{(1)}(x)$,

$$E\{[X_k^{(1)}(t) - a_k t]^2\} = \int x^2 dM_k^{(2)}(x)$$

and by the two-series theorem

$$\sum_k [X_k^{(1)}(t) - a_k t]$$

converges almost surely. A simple application of Doob's inequality yields in fact almost sure uniform convergence in finite time intervals. If we now reassemble the pieces we see that

$$E\{e^{i\,y\,X(t)}\} = \exp\left[t\int_{|x|\le 1}(e^{i\,y\,x} - 1 - i\,y\,x)dM(x) + \int_{|x|>1}(e^{i\,y\,x} - 1)dM(x)\right]$$

which is essentially the same as Levy-Khintchine representation for infinitely divisible distributions except for the Gaussian term that is missing. That is the continuous part, that cannot be built from jumps and needs the Brownian Motion or Wiener Process.

Chapter 2

Continuous time Markov Processes

2.1 Jump Markov Processes.

If we have a Markov Chain $\{X_n\}$ on a state space X, with transition probabilities $\Pi(x\,, dy)$, and a Poisson Process $N(t)$ with intensity λ, we can combine the two to define a continuous time Markov process $x(t)$ with X as state space by the formula

$$x(t) = X_{N(t)}$$

The transition probabilities of this Markov process are given by

$$\begin{aligned} P_x\{x(t) \in A\} &= P\{x(t) \in A | x(0) = x\} = p(t\,, x\,, A) \\ &= \sum_{k=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \Pi^{(k)}(x\,, A) \end{aligned}$$

Because of the Markov property of $\{X_n\}$ and the independence of the increments of $N(t)$ it is not difficult to see that

$$\begin{aligned} P\{x(t) \in A | \mathcal{F}_s\} &= P\{X_{N(t)-N(s)} \in A | X_0 = X_{N(s)}\} \\ &= P\{X_{N(t-s)} \in A | X_0 = x(s)\} \\ &= p(t - s\,, x(s)\,, A) \end{aligned}$$

thereby proving the Markov property of $x(t)$.

In general, if we want to define a Markov process on a state space X, we need the transition probabilities $p(t\,,x\,,A)$ defined for $t>0$, $x\in X$ and $A\in\mathcal{B}$, a σ- field of measurable subsets of X. It is natural to define $p(0\,,x\,,A)=\chi_A(x)$. For each $t>0$, $p(t\,,\cdot\,,\cdot)$ will be a transition probability from $X\to X$ and the collection will have to satisfy the Chapman-Kolmogorov equations

$$p(t+s\,,x\,,A)=\int_X p(s\,,y\,,A)\,p(t\,,x\,,dy)$$

Given such a collection, for each starting point $x\in X$, we can define a consistent family of finite dimensional distributions on the space Ω of trjectories $\{x(\cdot):[0\,,\infty)\to X\}$. On the σ-field $\mathcal{F}(t_1\cdots,t_k)$ corersponding to the times $t_1<\cdots<t_k$ we first define it for rectangles $[x(\cdot)\,:x(t_j)\in A_j\,,\forall j=1\cdots k\,]$

$$P_x\{\cap_{j=1}^k[x(t_j)\in A_j]\}=\int_{A_1}\cdots\int_{A_k}p(t_1\,,x\,,dy_1)\cdots p(t_k-t_{k-1}\,,x\,,dy_k)$$

The measure is then extended from rectangles to arbitrary sets in the product σ-fileld $\mathcal{F}(t_1\ \cdots,t_k)$. The consistency of these finite dimnsional distributions is a consequence of the Chapman-Kolmogorov property. Once we have consistency, a Theorem of A.Tulcea and I.Tulcea guarantees the existence of a measure P_x on the σ-field $\mathcal{F}$ generated by all the $\mathcal{F}(t_1\cdots,t_k)$. In our case using the fact that for any $n,m\ge 0$

$$\int_X \Pi^{(n)}(x\,,dy)\Pi^m(y\,,A)=\Pi^{(n+m)}(x\,,A)$$

we can deduce tha Chapman-Kolmogorov equations. The process itself lives on the space of step functions with values in X. The trajectories wait for an exponential waiting time with parameter λ and, if they are at x, jump to y with probability $\Pi(x\,,dy)$.

Remark 1. *If the Markov chain is a random walk on R with transition probability*

$$\Pi(x\,,A)=\alpha(A-x)$$

then the Markov Process is a compound Poisson process with a Levy measure $\lambda\,\alpha$.

2.2 Semigroups of Operators.

Any Markov transition function $p(t\,,x\,,dy)$ defines an operator

$$(T_t f)(x) = \int_X f(y)\,p(t\,,x\,,dy)$$

from the space $\mathbf{B}$ of bounded measurable functions on X into itself. $\{T_t\}$ has the following properties.

1. For each $t \geq 0$, T_t is a bounded linear operator from $\mathbf{B} \to \mathbf{B}$ and $\|T_t\| \leq 1$. In fact, because $T_t 1 = 1$, $\|T_t\| = 1$. $T_0 = I$, the identity operator and for any $t, s \geq 0$, $T_{t+s} = T_t T_s = T_s T_t$.

2. T_t maps non-negative functions into non-negative functions.

3. For any $t \geq 0$,

$$T_t = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k \pi^k}{k!}$$

where π is the operator

$$(\pi f)(x) = \int_X f(y)\Pi(x\,,dy)$$

It is clear that we can write

$$T_t = e^{-\lambda t}\exp[\lambda t \pi] = \exp[\big[\lambda t[\pi - I]\big] = \exp[tL]$$

where $L = \lambda[\pi - I]$.

Such families are called semigroups of operators and L is called the infinitesimal generator of the semigroup. In our case we can recover L from T_t by the formula

$$L = \lim_{t \downarrow 0} \frac{T_t - I}{t}$$

From the expansion of T_t in powers of π this relationship is easy to derive. It is valid in the strongest sense. As operators one can check that

$$\|\frac{T_t - I}{t} - L\| \to 0$$

as $t \downarrow 0$. The operator L contains all the relevent information regarding the Markov Process. In the case of Processes with independent increments with lots of small jumps (but with out centering)

$$(Lf)(x) = \int_R [f(x+y) - f(x)] M(dy)$$

where $M(dy)$ is an infinite measure that intgrates $|x|$ near 0. For L to be well defined for f we need some thing like a Lipshitz condition on f. Otherwise Lf may not be defined. In this case L is not a bounded operator and the expansion in powers of L is very suspect.

Although we have only looked at operators of the form

$$(Lf)(x) = \lambda \int_X [f(y) - f(x)] \Pi(x\,, dy)$$

as potential candidates of generators of Markov Processes one can see that operators of the form

$$(Lf)(x) = a(x) \int_X [f(y) - f(x)] \Pi(x\,, dy)$$

where $a(x)$ is a bounded non-negative function work just as well. It is not all that different. We pick $\lambda > 0$ such that $\lambda \geq a(x)$ for all x. Redefine

$$\hat{\Pi}(x\,, A) = \frac{a(x)}{\lambda} \Pi(x\,, A) + (1 - \frac{a(x)}{\lambda}) \chi_A(x)$$

so that

$$(Lf)(x) = a(x) \int_X [f(y) - f(x)] \Pi(x\,, dy) = \lambda \int_X [f(y) - f(x)] \hat{\Pi}(x\,, dy)$$

We have allowed for the possibility of not jumping at the exponential time with some probability and waiting for the next chance therby effectively reducing the jump rate, to different levels at different points.

If X is a finite set $\{1, \cdots, k\}$ then L has a matrix reperesentation

$$L = \{a_{i,j}\} \tag{1}$$

and if $p_{i,j}(t)$ are the transition probabilities at time t, one can check that for $i \neq j$

$$a_{i,j} = \lim_{t \downarrow 0} \frac{p_{i,j}(t)}{t} \geq 0 \tag{2}$$

and

$$a_{i,i} = \lim_{t \downarrow 0} \frac{p_{i,i}(t) - 1}{t} \leq 0$$

Also $\sum_j p_{i,j}(t) \equiv 1$ leads to

$$\sum_j a_{i,j} = 0 \quad \forall \quad i = 1, \cdots, k. \tag{3}$$

Such matrices have an interpretation that is important. The last property is simply the fact that $(L1)(x) \equiv 0$. If f is a function on the state space and $f(x_0)$ is the minimum of the function then $(T_t f)(x_0)$ being an average of f is atleast $f(x_0)$. The inequality $(T_t f)(x_0) \geq f(x_0)$ leads, upon differentiation at $t = 0$, to $(Lf)(x_0) \geq 0$. This is called the maximum principle (although we have stated it for the minimum). One can verify that for matrices $L = \{a_{i,j}\}$ the maximum principle is valid if and only if properties (1), (2), and (3) are valid.

The converse is valid as well. If L is a matrix satsfying properties (1), (2) and (3) then we can define $P(t) = \{p_{i,j}(t)\}$, by

$$P(t) = \exp[tL] = \sum_{k=0}^{\infty} e^{-t} \frac{(tL)^k}{k!}$$

and verify that $P(t)$ can infact serve as the transition probabilities of a Markov Process on the state space $X = \{1, \cdots, k.\}$. If $a_{i,j} > 0 \quad \forall\ i \neq j$, it is easy to see that $P(t)$ is a transition matrix provided t is small. But $P(t) = [P(\frac{t}{n})]^n$ and now $P(t)$ is seen to be a stochastic matrix for all $t > 0$. The case when $a_{i,j} \geq 0$, $\forall\, i \neq j$ is handled by approximating it by $\{a^{(n)}_{i,j}\}$ with $a^{(n)}_{i,j} > 0$, $\forall\, i \neq j$.

Exercise: If $P(t)$ are stochatic matrices that satisfy $P(t+s) = P(t)P(s) = P(s)P(t)$ for all $t, s > 0$ and $P(t) \to I$ as $t \to 0$, then show that

$$L = \lim_{t \downarrow 0} \frac{P(t) - I}{t}$$

exists and satisfies (1), (2) and (3).

Example: Birth and Death Processes. Imagine a population with a size that can be any nonnegative integer. The state space $X = Z^+ = \{n : n \geq 0\}$. If the current size of the population is n, either a birth or death can take place and change the population to $n \pm 1$. Let us denote the birth rate by λ_n and the death rate by μ_n. This is interpreted as

$$a_{n,n+1} = \lambda_n \quad \text{for} \quad n \geq 0$$

$$a_{n,n-1} = \mu_n \quad \text{for} \quad n \geq 1$$

and of course $\mu_0 = 0$ and

$$a_{n,n} = -(\lambda_n + \mu_n) \quad \text{for} \quad n \geq 0.$$

If each individual gives birth at rate λ independently of others, then $\lambda_n = n\lambda$. Similarly if the death rate is μ for each individual we have $\mu_n = n\mu$. For a queue with a single server where the arrival time and the service time are exponetial $\lambda_n = \lambda$ for $n \geq 0$ and $\mu_n = \mu$ for $n \geq 1$.

2.3 Markov Processes and Martingales.

Suppose L is the bounded operator

$$(Lf)(x) = a(x) \int_X [f(y) - f(x)] \Pi(x\,, dy)$$

where $a(x)$ is a bounded measurable function and Π is a transition probability on X. Let $T(t) = \exp[t\,L]$ be the operators

$$(T_t f)(x) = \int_X f(y)\, p(t\,, x\,, dy)$$

where $p(t\,, x\,, dy)$ are the transition probabilities corresponding to L. We swa earlier by differentiating term by term in the expansion

$$T_t = \sum_{k=0}^{\infty} \frac{(t\,L)^k}{k!}$$

of T_t in powers of L that

$$\frac{d\,T_t}{dt} = T_t\,L = L\,T_t$$

In particular

$$T_t\, f - f = \int_0^t T_s\, L\, f\, ds$$

or

$$E_x\Big\{ f(x(t)) - f(x(0)) - \int_0^t (Lf)(x(s))\, ds \Big| \mathcal{F}_0 \Big\} = 0$$

If we use the homogeniety in time

$$E_x\Big\{ f(x(t)) - f(x(s)) - \int_s^t (Lf)(x(\sigma))\, d\sigma \Big| \mathcal{F}_s \Big\} = 0$$

Thus

$$M_f(t) = f(x(t)) - f(x(0)) - \int_0^t (Lf)(x(\sigma))\, d\sigma$$

is a Martingale with respect to any Markov process P_x with generator L starting from an arbitrary point $x \in X$. By a similar argument one can show that for any λ,

$$e^{-\lambda\, t} f(x(t)) - f(x(0)) - \int_0^t e^{-\lambda\,\sigma}[(Lf) - \lambda\, f](x(\sigma))\, d\sigma$$

is again a martingale. If A is ameasurable set then

$$\tau_A = \{\inf t : x(t) \in A\}$$

is seen to be a stopping time and if $\tau(\omega)$ is finite then $x(\tau) \in A$. Note

$$\{\tau_A \le t\} = \bigcup_{\substack{s \le t \\ s \text{ rational}}} \{x(s) \in A\} \bigcup \{x(t) \in A\}$$

Lemma 2.1. *For $\lambda > 0$ the function*

$$U(x) = E_x\big\{e^{-\lambda\,\tau_A} f(x(\tau_A))\big\}$$

is the unique solution of

$$(LU)(x) = \lambda U(x) \quad \text{for} \quad x \in A^c$$

with $U(x) = f(x)$ on A.

Proof. Clearly for $x \in A$, $\tau_A = 0$ a.e. P_x and $U(x) = f(x)$. For $x \notin A$, starting from

$$U(x) = E_x\{e^{-\lambda \tau_A} f(x(\tau))\}$$

and conditioning with respect to $\mathcal{F}_\tau$ where

$$\tau = \{\inf t : x(t) \neq x(0)\}$$

we get

$$\begin{aligned} U(x) &= E_x\{e^{-\lambda\tau} E_{x(\tau)}\{e^{-\lambda\tau_A} f(x(\tau_A))\} \\ &= E_x\{e^{-\lambda\tau} U(x(\tau))\} \\ &= E_x\{e^{-\lambda\tau}\} E_x\{U(x(\tau))\} \\ &= \frac{a(x)}{\lambda + a(x)} \int_X U(y)\Pi(x\,, dy) \end{aligned}$$

We can rewrite this as

$$(\lambda + a(x))U(x) = a(x) \int_X U(y)\Pi(x\,, dy)$$

or

$$(LU)(x) = a(x) \int_X [U(y) - U(x)]\Pi(x\,, dy) = \lambda U(x).$$

Conversely if U solves $LU = \lambda U$ on A^c with $U = f$ on A, we know that

$$e^{-\lambda t} U(x(t)) - U(x(0)) - \int_0^t [LU - \lambda U](x(s))\, ds$$

is a P_x martingale. We apply Doob's stopping theorem with the stopping time τ_A. Because $(LU)(x(s) - U(x(s)) = 0$ for $0 \leq s \leq \tau_A$ and $U(x(\tau_A)) = f(x(\tau_A))$, we conclude that

$$U(x) = E_x\{e^{-\lambda\tau_A} f(x(\tau_A))\}$$

proving uniqueness. ☐

There are some other related equations which have unuique solutions expressible in terms of the process. For instance

$$\lambda U - LU = g \;\text{ for }\; x \in A$$

with $U = f$ for $x \in A^c$ can be solved uniquely and the solution is given by

$$U(x) = E_x\left\{e^{-\lambda\tau_A} f(x(\tau_A)) + \int_0^{\tau_A} e^{-\lambda s} g(x(s))\, ds\right\}$$

To solve the equation

$$LU = 0 \ \text{ for } \ x \in A$$

uniquely with $U = f$ for $x \in A^c$ we require the assumption

$$P_x\{\tau_A < \infty\} = 1$$

for $x \in A$ and to solve

$$LU = g \ \text{ for } \ x \in A$$

uniquely with $U = f$ for $x \in A^c$ we need the assumption

$$E_x\{\tau_A\} < \infty$$

for $x \in A$.

2.4 Explosion, Recurrence and Transience.

Let us consider a simple birth process where $\lambda_n = (n+1)^2$ and $\mu_n = 0$. If the process starts at 0, it moves successively through $1, 2, 3, \cdots$ waiting for an exponential time τ_n at n with $E\{\tau_n\} = \frac{1}{(n+1)^2}$ for $n \geq 0$. A simple calculation shows that

$$\sum_{n=0}^{\infty} E\{\tau_n\} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} < \infty$$

and by Fubini's theorem

$$\sum_{n=0}^{\infty} \tau_n = \zeta < \infty$$

a.e. The process will reach ∞ at time ζ which is finite a.e. We say that explosion takes place in finite time with probability 1. This can happen with any Markov Process that we want to construct with

$$(Lf)(x) = a(x) \int_X [f(y) - f(x)]\Pi\,(x\,, dy)$$

if the function $a(\cdot)$ is unbounded. If ζ is the explosion time i.e. the first time when we have an infinite number of jumps, the process is well defined upto that time by the canonical construction. If $\sigma > 0$ is any constant, we define

$$U_\sigma(x) = E_x\{e^{-\sigma\zeta}\}$$

Then $0 \leq U(x) < 1$ for each $x \in X$ and there is no explosion if and only if $U(x) \equiv 0$ on X. If τ_1 is the time of the first jump then $U_\sigma(\cdot)$ satisfies

$$\begin{aligned} U_\sigma(x) &= E_x\Big\{e^{-\sigma\tau_1}U(x(\tau_1))\Big\} \\ &= \frac{a(x)}{a(x)+\sigma}\int_X U_\sigma(y)\Pi\,(x\,,dy) \end{aligned}$$

This can be rewritten as

$$(LU_\sigma)(x) = \sigma U_\sigma(x) \tag{1}$$

which is valid for every $x \in X$. Conversely if $U(\cdot)$ is any solution of (1) with $0 \leq U(x) \leq 1$ then

$$\begin{aligned} U(x) &= E_x\Big\{e^{-\sigma\tau}U(x(\tau))\Big\} \\ &= E_x\Big\{e^{-\sigma\tau_1}E_{x(\tau_1)}\Big\{e^{-\sigma\tau_1}U(x(\tau_1))\Big\}\Big\} \\ &= E_x\Big\{e^{-\sigma\tau_2}U(x(\tau_2))\Big\} \\ &\cdots\cdots\cdots \\ &= E_x\Big\{e^{-\sigma\tau_n}U(x(\tau_n))\Big\} \end{aligned}$$

where τ_n is the time of the n-th jump. Therefore

$$U(x) \geq E_x\{e^{-\sigma\tau_n}\}$$

for every n and letting $n \to \infty$

$$U(x) \geq E_x\{e^{-\sigma\zeta}\}$$

Explosion is equivalent to the existence of a nonnegative bounded solution of (1). We can always normalize to get a function that satisfies $0 \leq U(x) \leq 1$. We do not really need a solution.

$$U(x) \leq E_x \Big\{ e^{-\sigma\tau} U(x(\tau)) \Big\}$$

will suffice instead of

$$U(x) = E_x \Big\{ e^{-\sigma\tau} U(x(\tau)) \Big\}$$

because this will lead to the inequlity

$$U(x) \leq E_x \Big\{ e^{-\sigma\tau_n} U(x(\tau_n)) \Big\} \leq E_x \big\{ e^{-\sigma\tau_n} \big\}$$

which is just as good. So a necessary and sufficient condition for explosion is the existence of a nonzero function $U(\cdot)$ with $0 \leq U(x) \leq 1$ for all $x \in X$ that satisfies

$$(LU)(x) \geq \sigma U(x)$$

for some $\sigma > 0$. The conditions are equivalent for different values of $\sigma > 0$. The nonexistence of a function U is harder to check. If there is a nonnegative function $U(\cdot)$ on X, that satisfies

$$(LU)(x) \leq \sigma U(x)$$

for some $\sigma > 0$ and $U(x) \to \infty$ whenever $a(x) \to \infty$ then we do not have explosion. To see this we note that

$$U(x) \geq E_x \Big\{ e^{-\sigma\tau_n} U(x(\tau_n)) \Big\}$$

We know that when $n \to \infty$ either $\tau_n \to \infty$ or $a(x(\tau_n)) \to \infty$ almost surely. By our assumption $U(x(\tau_n)) \to \infty$ whenever $a(x(\tau_n)) \to \infty$. But the bound does not allow $U(x(\tau_n)) \to \infty$ while τ_n remains bounded.

In any case if L is unbounded we canot use the expansion

$$T_t = \sum_{j=0}^{\infty} \frac{(tL)^j}{j!}$$

and if there is explosion T_t is not well determined by L i.e by $a(\cdot)$ and $\Pi\,,(x\,dy)$.

We will concentrate on the class of birth death processes to illustrate these ideas.

For any birth and death process we can solve the equation

$$(LU_\sigma)(i) = \lambda_i U_\sigma(i+1) + \mu_i U_\sigma(i-1) - (\lambda_i + \mu_i)U_\sigma(i) = \sigma U_\sigma(i) \qquad (3)$$

for $i = 0, 1, 2, \cdots, n-1$ with boundary condition $U_\sigma(n) = 1$. Notice that because we must have $\mu_0 = 0$,

$$\lambda_0 U_\sigma(1) = (\lambda_0 + \sigma)U_\sigma(0)$$

or

$$U_\sigma(1) = \frac{\lambda_0 + \sigma}{\lambda_0} U_\sigma(0) \qquad (4)$$

From what we proved in the last section the solution $U_{\sigma,n}$ of (2) and (3) satisfies

$$U_{\sigma,n}(0) = E_0\{e^{-\sigma\tau_n}\}$$

For the validity of this formula we need only the values of $\{\lambda_j,\, \mu_j\}$ for $0 \le j \le n-1$. We can define them arbitraily for $j \ge n$ and so it does not matter if the sequence $\{\lambda_j\,, \mu_j\}$ is unbounded. On the other hand if $U_\sigma(\cdot)$ is any solution of (2) and (3), so is

$$U_{\sigma,n}(i) = \frac{U_\sigma(i)}{U_\sigma(n)}$$

and $U_{\sigma,n}(n) = 1$. Hence

$$U_{\sigma,n}(i) = E_i\{e^{-\tau_n}\} \;\text{ for }\; 0 \le i \le n$$

Clearly $U_{\sigma,n}(i)$ is $\downarrow$ in n for $n \ge i$ and nonexplosion is equivalent to

$$\lim_{n\to\infty} U_{\sigma,n}(i) = 0$$

for each fixed i. Equivalently if we solve (2) and (3) for all $i \ge 0$, to get $U(\cdot)$ we need

$$\lim_{n\to\infty} U_\sigma(n) = \infty$$

For determinig $U_\sigma(n)$ we can assume with out loss of generality that $U_\sigma(0) = 1$. Then $U_\sigma(1) = \frac{\lambda_0+1}{\lambda_0}$ and the recurrence relation

$$U_\sigma(i+1) = \frac{(\lambda_i + \mu_i + \sigma)U_\sigma(i) - \mu_i U_\sigma(i-1)}{\lambda_i} \tag{5}$$

determines $U_\sigma(n)$ successively for all $n \geq 2$. The $U_\sigma(n)$ that we get is necessairily increasing and explosion occurs if and only if

$$\lim_{n\to\infty} U_\sigma(n) < \infty$$

We can rewrite equation (4) in the form

$$U_\sigma(i+1) - U_\sigma(i) = \frac{\mu_i}{\lambda_i}[U_\sigma(i) - U_\sigma(i-1)] + \frac{\sigma}{\lambda_i}U_\sigma(i) \tag{6}$$

for $i \geq 1$ and

$$U_\sigma(1) = \frac{\lambda_0 + \sigma}{\lambda_0}U_\sigma(0) = \frac{\lambda_0 + \sigma}{\lambda_0} = 1 + \frac{\sigma}{\lambda_0}$$

with a normalization of $U_\sigma(0) = 1$ Clearly $U_\sigma(i)$ is increasing as a function of i and $U_\sigma(i) \geq 1$ for all i. For $U_\sigma(i)$ to remain bounded it is therefore necessary that the solution of

$$V_\sigma(i+1) - V_\sigma(i) = \frac{\mu_i}{\lambda_i}[V_\sigma(i) - V_\sigma(i-1)] + \frac{\sigma}{\lambda_i} \tag{7}$$

for $i \geq 2$ with $V_\sigma(1) = 1 + \frac{\sigma}{\lambda_0}$ and $V_\sigma(0) = 1$ should be bounded. Since constants are solutions of the homogeneous equation and the solutions are linear in the coefficient of the inhomogeneous term σ, there is basically one equation that needs to have a bounded solution.

$$U(i+1) - U(i) = \frac{\mu_i}{\lambda_i}[U(i) - U_\sigma(i-1)] + \frac{1}{\lambda_i} \tag{8}$$

for $i \geq 2$ with $U(0) = 0$ and $U(1) = 1$. The converse is true as well. If equation (8) has a solution bounded by C we find that, replacing $U(\cdot)$ by $V(\cdot) = U(\cdot) + 1$ which is bounded by $C + 1$,

$$V(i+1) - V(i) \geq \frac{\mu_i}{\lambda_i}[V(i) - V(i-1)] + \frac{1}{\lambda_i}\frac{V(i)}{C+1} \tag{9}$$

with $V(0) = 1$ and $V(1) = 2$. By comparison we conclude that if $\sigma_0 = \min(\lambda_0\,, \frac{1}{C+1})$

$$U_{\sigma_0}(i) \le V(i)$$

Although we know that boundedness of $U_\sigma(\cdot)$ for some value of $\sigma > 0$ implies the same for all other positive values of σ, we can see it directly by the following simple inequality, which is a consequence of Jensen's inequality. For any convex function ϕ

$$(L\phi(u))(x) \ge \phi'(u(x))(Lu)(x)$$

In particular if

$$V(i+1) - V(i) \ge \frac{\mu_i}{\lambda_i}[V(i) - V(i-1)] + \frac{\sigma}{\lambda_i}V(i)$$

then for $\alpha > 1$, $V_\alpha(i) = [V(i)]^\alpha$ satisfies

$$V_\alpha(i+1) - V_\alpha(i) \ge \frac{\mu_i}{\lambda_i}[V_\alpha(i) - V_\alpha(i-1)] + \frac{\sigma\alpha}{\lambda_i}V(i)$$

establishing for $\alpha > 1$, and $\sigma > 0$

$$U_\sigma(i) \le U_{\alpha\,\sigma}(i) \le [U_\sigma(i)]^\alpha$$

Now one can write down necessary and sufficient conditions for the existence of a bounded solution to (8) as

$$\sum_{n=1}^{\infty} \frac{\mu_1\mu_2\cdots\mu_n}{\lambda_1\lambda_2\cdots\lambda_n} \sum_{j=1}^{n} \frac{\lambda_1\lambda_2\cdots\lambda_{j-1}}{\mu_1\mu_2\cdots\mu_{j-1}}\frac{1}{\mu_j} < \infty$$

and

$$\sum_{n=1}^{\infty} \frac{\mu_1\mu_2\cdots\mu_n}{\lambda_1\lambda_2\cdots\lambda_n} < \infty$$

Remark 2. *In fact the probability of explosion in a finite time is either identically* 1 *for all initial states or identically equal to* 0 *for all states.*

To see this

$$P_i\{\zeta < \infty\} \ge E_i\{e^{-\zeta}\} = \lim_{n\to\infty} E_i\{e^{-\tau_n}\} = \lim_{n\to\infty}\frac{U(i)}{U(n)} = \frac{U(i)}{U(\infty)}$$

since $\lim_{n\to\infty} U(n) = U(\infty)$ exists. Now we see that

$$\lim_{i\to\infty} V(i) = 1$$

where

$$V(i) = P_i\{\zeta < \infty\}$$

On the other hand one can verify easily that

$$(LV)(i) = 0$$

for all i (including 0), and the only solution to this are the constants. Therefore $V(i) \equiv c$ and of course $c = 1$, if $U(\infty) < \infty$. Otherwise $V(i) \equiv 0$.

Example: For a birth and death process with $\lambda_n - \mu_n \leq C(n+1)$ for all n, there is no explosion. Let us try $U(n) = (n+1)^\alpha$ with $\alpha < 1$. Since $U(\cdot)$ is concave

$$\begin{aligned}(LU)(n) &= \lambda_n(U(n+1) - U(n)) + \mu_n(U(n-1) - U(n)) \\ &\leq (\lambda_n - \mu_n)(U(n) - U(n-1)) \\ &\leq C\alpha(n+1)(n+1)^{\alpha-1} \\ &= C\alpha U(n)\end{aligned}$$

Recurrence and Transience.

For a birth and death process we have essentially investigated the solutions of the equation

$$\mu_n U(n-1) - (\lambda_n + \mu_n)U(n) + \lambda_n U(n+1) = 0$$

for $n \geq 1$. If we normalize $U(1) - U(0) = 1$, we obtain

$$U(n+1) - U(n) = \Pi_{j=1}^n \frac{\mu_j}{\lambda_j}$$

and with $U(0) = 0$

$$U(n+1) = 1 + \sum_{k=1}^n \Pi_{j=1}^k \frac{\mu_j}{\lambda_j}$$

It is easy to see that $U(n)$ becomes unbounded as $n \to \infty$ if and only if

$$\sum_{k=1}^{\infty} \Pi_{j=1}^{k} \frac{\mu_j}{\lambda_j} = \infty$$

Now we return to the question of recurrence. Given three states $0 < i < n$ we try to calculate the probaility

$$P_i\{\tau_n < \tau_0\}$$

of visiting n before visiting 0 when the process starts from i. Here τ_j is the first visiting time of the state j. This probability is defined well before explosion and we can alter λ_k , μ_k for $k \geq n$ with out changing the probability. In fact we can assume without any loss of generality that L is a bounded operator. If $LU(i) = 0$ for $0 \leq i \leq n$, then $U(x(\tau_n \wedge t))$ is seen to be a martingale because we can always modify L, U and g beyond n. In particular if $U(0) = 0$, by Doob's theorem,

$$P_i\{\tau_n < \tau_0\} = \frac{U(i)}{U(n)}$$

and there is recurrence if and only if $U(n) \to \infty$. Actually we do not have to solve the equation $LU = 0$. If we can find a supersolution $LU \leq 0$ which goes to ∞ as $n \to \infty$ that works just as well to prove recurrence and the existence of a positive subsolution that remains bounded as $n \to \infty$ will establish transience. In particular the necessary and sufficient condition for recurrence, with nonexplosion, is

$$\sum_{n=1}^{\infty} \frac{\mu_1 \cdots \mu_n}{\lambda_1 \cdots \lambda_n} = \infty$$

The question of recurrence with explosion raises the philosophical question of life after death.

Exercise: For a birth and death process, if $\mu_n - \lambda_n \leq 0$ for $n \geq n_0$, then the process is recurrent.

Invariant distributions and positive recurrence.

An invariant distribution $q(dy)$ on $(X \,, \mathcal{B})$ is a pobability distribution such that

$$q(A) = \int_X p(t \,, x \,, A) \, q(dy)$$

for all $A \in \mathcal{B}$ and $t > 0$. If L is bounded this can be verified by verifying the infinitesimal relation

$$\int_X (Lf)(x) q(dx) = 0$$

for all $f \in \mathbf{B}$. To see this we only have to observe that

$$\frac{d}{dt} < T_t f \, , q > = < L T_t f \, , q >= 0.$$

Or equivalently it is enough to verify that

$$(L^* q)(A) = \int_X [\Pi\,(y\,,A) - \chi_A(y)] a(y) q(dy) = 0$$

In general the invariant distribution is not unique. In the case of a birth and death process this reduces to

$$q(i-1)\lambda_{i-1} + q(i+1)\mu_{i+1} = q(i)(\lambda_i + \mu_i)$$

for $i \geq 1$, with

$$q(1)\mu_1 = q(0)\lambda_0 \quad \text{or} \quad \frac{q(1)}{q(0)} = \frac{\lambda_0}{\mu_1}$$

We can assume for the momemt that $q(0) = 1$. Then

$$q(i) = \frac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i}$$

satisfies

$$\begin{aligned} q(i-1)\lambda_{i-1} + q(i+1)\mu_{i+1} &= q(i-1)[\lambda_{i-1} + \frac{\lambda_{i-1}\lambda_i}{\mu_i}] \\ &= q(i-1)\frac{\lambda_{i-1}}{\mu_i}(\lambda_i + \mu_i) \\ &= q(i)(\lambda_i + \mu_i) \end{aligned}$$

or

$$q(i) = \frac{\lambda_{i-1}}{\mu_i} q(i-1)$$

The convergence of the series

$$Z = \sum_{i=0}^{\infty} q(i) = 1 + \sum_{i=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i} < \infty \tag{10}$$

is necessary and sufficient for the existence of a finite invariant measure and we can normalize it so that $\frac{1}{Z}q(i)$ is the invariant probability distribution. While it is clear that when L is bounded or when $\{\lambda_n\,,\mu_n\}$ is bounded $q(\cdot)$, constructed under the assumption (10), is really an invariant probability for the process. This is not obvious in the unbounded case. It is however true in the absence of explosion.

The proof is not very hard. We can truncate the system at size n by redefining $\lambda_n = 0$ and then the finite system has an invariant measure

$$q_n(i) = \frac{1}{Z_n}\frac{\lambda_0\lambda_1\cdots\lambda_{i-1}}{\mu_1\mu_2\cdots\mu_i}$$

with

$$Z_n = 1 + \sum_{j=1}^{n}\frac{\lambda_0\lambda_1\cdots\lambda_{i-1}}{\mu_1\mu_2\cdots\mu_i}$$

If we let $n \to \infty$ the absence of explotion yields

$$p^{(n)}_{i,j}(t) \to p_{i,j}(t)$$

as $n \to \infty$ for all i, j and $t > 0$. The invariance of $q_n(\cdot)$ for the truncated process carries over to the limit.

Beyond Explosion:

The question of how to define the process is upto us. Anything reasonable will give us a process. For instance at the explosion time we can reinitiate at the state 0. Start afresh with a clean slate. Now the process is well defined for all times. One may see several explosions and each time we restart from 0. By the time we run out of luck again, i.e. when we see an infinite number of explosions, real time would have become infinite. In terms of analysis this corresponds to 'boundary conditions' that U has to satisfy for the formal relation

$$(LU)(i) = f(i)$$

to be really valid. In the case we just discussed, it is

$$\lim_{i\to\infty} U(i) = U(0)$$

It is as if we are really on a circle and as $i \to \infty$ it approaches 0 from the other side. For each rule of how to continue after explosion there will be a different boundary condition.

The proofs are left as exercises.