Problemset 7.

Q1. Give an example of a sequence $f_n(x)$ that converges weakly to $f(x) \equiv 1$ on $L_2[0,1]$ with Lebesgue measure, but not strongly. (in the norm)

Q2. A family of continuous functions $f_\alpha(x)$ on $[0,\infty)$ satisfy

$$\lim_{\delta \to 0} \sup_{\alpha} \sup_{\substack{|x-y| \le \delta \\ x,y \le N}} |f_\alpha(x) - f_\alpha(y)| = 0$$

for every $N < \infty$,

$$\sup_{x \ge 0} \sup_{\alpha} |f_\alpha(x)| \le C < \infty$$

$$\lim_{N \to \infty} \sup_{x \ge N} \sup_{\alpha} |f_\alpha(x)| = 0$$

Show that every sequence from the family has a convergent subsequence in the Banach space $\mathcal{X} = C[0,\infty)$ with the norm $\|f - g\| = \sup_{x \ge 0} |f(x) - g(x)|$.

Q3. $K(s,t)$ is a bounded continuous function of two variables on $[0,\infty) \times [0,\infty)$.

$$(Tf)(s) = \int_0^\infty K(s,t) e^{-(s+t)} f(t)\, dt$$

Show that T is a compact operator from $\mathcal{X}$ to $\mathcal{X}$.