Assignment 9.

Problem 1. Denote by $f = \frac{d\alpha}{d\lambda}$. Then since $\alpha(A) \le \lambda(A)$ for all $A \in \Sigma$, we have $0 \le f(x) \le 1$ a.e. Define

$$A = \{x : f(x) = 0\}, B = \{x : 0 < f(x) < 1\}, C = \{x : f(x) = 1\}$$

Since $\alpha(A) = \int_A f(x) d\lambda$ and $f(x) = 0$ a.e. on A, clearly $\alpha(A) = 0$. Since $\alpha + \beta = \lambda$ it is clear that $\frac{d\beta}{d\lambda} = g(x) = 1 - \frac{d\alpha}{d\lambda} = 1 - f(x) = 0$ on C and $\beta(C) = 0$. On the other hand on B,

$$\frac{d\alpha}{d\lambda} = f(x), \quad \frac{d\beta}{d\lambda} = 1 - f(x)$$

are both non-zero, hence

$$\frac{d\alpha}{d\beta} = \frac{f(x)}{1 - f(x)}, \quad \frac{d\beta}{d\alpha} = \frac{1 - f(x)}{f(x)}$$

which implies that $\alpha << \beta$ and $\beta << \alpha$.

Problem 2. If $f_n(x) \to a$ in $L_2(\alpha)$ then $f_n(x) \to a$ in measure with respect to α and it then has a subsequence $f_{n_j}(x) \to a$ a.e with respect to α. That subsequence still converges to b in $L_2(\beta)$ and by a similar argument has a subsequence converging to b a.e β. We end up with the same subsequence converging to a a.e. α, and b a.e. β. If we denote the subsequence by g_j then with

$$A = \{x : \lim_{j\to\infty} g_j(x) = a\}, \quad B = \{x : \lim_{j\to\infty} g_j(x) = b\}$$

we have $A \cap B = \emptyset$ and $\alpha(A^c) = \beta(B^c) = 0$.

Problem 2. This is a simple calculation.

$$\int x_i d\alpha = a$$

$$\int f_n(x) d\alpha = a$$

If $i \ne j$,

$$\int x_i x_j d\alpha = a^2, \quad \int (x_i - a)(x_j - a) d\alpha = 0$$

and

$$\int x_i^2 d\alpha = a, \quad \int (x_i - a)^2 d\alpha = a(1 - a)$$

$$
\begin{aligned}
\int [f_n(x) - a]^2 \; d\alpha &= \frac{1}{n^2}\left[\sum_{i=1}^{n} \int (x_i - a)^2 d\alpha + \sum_{i,j:i\neq j} \int (x_i - a)(x_j - a) d\alpha\right] \\
&= \frac{1}{n^2}[na(1-a)] \\
&= \frac{a(1-a)}{n} \\
&\to 0 \quad \text{as } n \to \infty
\end{aligned}
$$

Similarly $\int [f_n(x) - b]^2 d\beta \to 0$. Now use Problem 1.