Real Variables Fall 2007.

Assignment 9. Due Nov 5.

Problem 1. Let α, β be two countably additive non-negative finite measures on (X, Σ). Show that there are disjoint sets $A, B.C$ in Σ with $X = A \cup B \cup C$ that satisfy

$\alpha(A) = 0$, $\beta(B) = 0$ and on C, $\alpha << \beta$ as well as $\beta << \alpha$. Show that the sets A, B, C are unique upto λ measure 0. (Hint: Start with $\lambda = \alpha + \beta$ and examine $\frac{d\alpha}{d\lambda}$)

Problem 2. If α and β are said to be orthogonal if there are disjoint sets A, B such that $\alpha(A^c) = \beta(B^c) = 0$. It is written $\alpha \perp \beta$. If there exists a sequence of measurable functions $\{f_n(x)\}$ on (X, Σ) such that $\|f_n(x) - a\|_{L_p(\alpha)} \to 0$ and $\|f_n(x) - b\|_{L_p(\beta)} \to 0$ where α, β are two finite measures and $a \neq b$ are two different constants then show that $\alpha \perp \beta$.

Problem 3. Let $X = \prod_{i=1}^{\infty} X_i$ where each X_i is the space $\{0, 1\}$ of two points. α, β are two product measures on (X, Σ) with $\alpha = \prod \alpha_i$ and $\beta = \prod \beta_i$. The $\{\alpha_i\}$ are all equal to the measure with $\alpha_i\{1\} = a, \alpha_i\{0\} = 1 - a$ where as $\{\beta_i\}$ are all equal with $\beta_i\{1\} = b, \beta_i\{0\} = 1 - b$. $0 < a, b < 1$ and $a \neq b$. Show that $\alpha \perp \beta$. (Hint: consider $f_n(x) = \frac{x_1 + \cdots + x_n}{n}$ and use Problem 2, with $p = 2$.)