Real Variables Fall 2007.

Assignment 8. Due Oct 29.

Problem. Let $\{x_j : j = 1, 2, \ldots\}$ be a countable set of points in $[0, 1]$ and $\{p_j\}$ a set of nonnegative numbers with $s = \sum_j p_j < \infty$. Define

$$F(x) = \sum_{j : x_j \le x} p_j$$

Of course $F(x) \equiv 0$ for $x < 0$ and $F(x) \equiv s$ for $x > 1$. Show that for almost all x with respect to Lebesgue measure

$$\limsup_{h \to 0} \frac{F(x+h) - F(x)}{h} = 0$$

Hint. Step 1. Let $A = \{x : \limsup_{h \to 0} \frac{F(x+h)-F(x)}{h} \ge q\}$. Then use a Vitali covering argument to show that for $b > 1$ and $a < 0$

$$F(b) - F(a) \ge \mu(A)\, q$$

Hint. Step 2. Separate the first n jumps so that

$$F(x) = F_1(x) + F_2(x)$$

with $F_2(x)$ having only a finite number of jumps. In particular

$$\lim_{h \to 0} \frac{F_2(x+h) - F_2(x)}{h} = 0$$

if there is no jump at x.

Hint. Step 3. Use step 1 to conclude that for any $\epsilon > 0$, by removing enough jumps one can make

$$\mu(A)\, q \le F_1(b) - F_1(a) \le \epsilon$$

Now finish up.