If $X$ is a Banach Space, because it is a complete metric space Baire category theorem applies. If $X = \cup_{i=1}^{\infty} C_i$ is a countable union of closed sets at least one of them will have interior. If $\{C_i\}$ are not closed then at lest one of them will have a closure that will have a nonempty interior. There are three applications we have in mind.

**1. Uniform Boundedness principle.** If $T_n : X \to Y$ are bounded linear maps from $X$ to another Banach space $Y$ and if for each $x \in X$

$$\sup_n \|T_n x\| = C(x) < \infty$$

then

$$\sup_n \sup_{\|x\| \le 1} \|T_n x\| = \sup_n \|T_n\| < \infty$$

Proof: Write

$$X = \cup_{\ell=1}^{\infty} \{x : \sup_n \|T_n x\| \le \ell\} = \cup_{\ell=1}^{\infty} C_\ell$$

Clearly each $C_\ell = \{x : \sup_n \|T_n x\| \le \ell\}$ is closed. By the Baire category theorem one of them has an interior. Let

$$\{y : \|y - y_0\| \le \delta_0\} \subset C_{\ell_0}$$

for some $y_0, \delta_0$ and $\ell_0$. Then if $\|z\| \le \delta_0$,

$$\|T_n z\| = \|T_n(y_0 + z) - T_n y_0\| \le \|T_n(y_0 + z)\| + \|T_n y_0\| \le 2\ell_0$$

Therefore

$$\sup_n \|T_n\| = \sup_n \sup_{\|x\| \le 1} \|T_n x\| \le \frac{2\ell_0}{\delta_0}$$

**2. Open Mapping Theorem.** Let $T : X \to Y$ be bounded linear map that maps $X$ **onto** $Y$. Then the image $TG$ of any open set $G \subset X$ is open in $Y$.

**Proof:** Let $B_\ell$ be the ball of radius $\ell$ in $X$ and $D_\ell = TB_\ell$ its image in $Y$. Since $T$ is onto $Y = \cup_{\ell=1}^{\infty} D_\ell$ and by Baire category theorem at least one of the $D_\ell$ say $D_{\ell_0}$ will have a closure $C_{\ell_0}$ with non empty interior, i.e it will contain a ball $\{y : \|y - y_0\| < \delta_0\}$. Since the difference of two balls of radius $\ell_0$ is contained in a ball of radius $2\ell_0$ it is clear that the closure of the image of a ball of radius $2\ell_0$ will contain a ball of radius $\delta_0$ around 0 in $Y$. By scaling, the closure of the image of a ball of radius $C\epsilon$ in $X$ will contain a ball of radius $\epsilon$ with $C = \frac{2\ell_0}{\delta_0}$ in $Y$. Let us prove that the image of a ball of radius $2C$ contains the ball of radius 1. Given $y$ wuith $\|y\| < 1$ we can find $x_1$ such that $\|x_1\| \le C$ and $\|Tx_1 - y\| < \frac{1}{2}$. With $y_1 = (y - Tx_1)$ we can find $x_2$ such that $\|x_2\| \le \frac{C}{2}$ and $\|Tx_2 - y_1\| \le \frac{1}{4}$ and so on. The sequence $x_n$ will have the property $\|x_n\| \le \frac{C}{2^n}$ and $T(\sum_i x_i) = y$. The completeness is needed to sum $\sum_i x_i$. The image of a big ball contains the unit ball. Therefore the image of any ball which is obtained from the image of the unit ball by dilation and translation will contain a ball. This is the open mapping theorem.

**Corollary.** If $T$ is bonded, one to one and onto its inverse is bounded.

**Proof:** If the image of a ball of radius $C$ contains the unit ball the inverse if it exists is bounded by $C$.

**Corollary.** If $\|x\|$ and $|||x|||$ are two complete norms on $X$ such that $\|x\| \leq C|||x|||$ then for some other $C'$ $|||x||| \leq C'\|x\|$

**Proof:** Think of $X$ under $|||x|||$ and $\|x\|$ as $X$ and $Y$ and the identity map as $T$. It is one to one, bounded and onto. Hence the inverse which is identity going in the other direction is bounded.

**3. Closed Graph Theorem.** Let $T$ be a linear map from $X \to Y$. Its graph is the set $\{(x, Tx) : x \in X\} \subset X \times Y$. It is a linear subset of $X \times Y$ which is a Banach space with norm $\|x\| + \|y\|$. The graph is closed if the set $\{(x, Tx) : x \in X\}$ is cosed as a subspace of $X \times Y$. If the graph is closed then $T$ is bounded. The graph being closed amounts to: if $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

**Proof:** Consider the map $S$ from the graph $Z \subset X \times Y$ to $X$ that sends $(x, Tx) \to x$. It is clearly bounded by 1 because $\|x\| \leq \|x\| + \|y\|$. It is onto. $Z$ is a closed subspace of a Banach space and so is a Banach space. Hence the inverse of $S$ that sends $x \to (x, Tx)$ is bounded, i.e $T$ is bounded.