Answers to Assignment 10.

Problem 1. $d(x, A) = 0$ if and only if there exists $y_n \in A$ such that $d(x, y_n) \to 0$, i.e., if and only if $x \in \bar{A}$ or if A is closed if and only if $x \in A$.

Problem 2. If $A \subset X$ and x_n is Cauchy in A, it is Cauchy in X and if X is complete converges to $x \in X$. But if A is closed then $x \in A$ and $x_n \to x$ in A, making A complete. On the other hand if A is not closed there is a sequence $x_n \in A$ suh that $x_n \to x \notin A$. Such a sequence will be Cauchy in A but will not have a limit in A. So A can not be complete. On the other hand if we change the metric from d to D if $x_n \in G$ and $x_n \to x \in G$, then $d(x_n, G^c)$ remains bounded away from 0 and $d(x_n, G^c) \to d(x, G^c)$. Hence $D(x_n, x) \to 0$. If $D(x_n.x) \to 0$ then clearly $d(x_n, x) \to 0$. The two metrics are therefore equivalent on G. To see that (G, D) is complete if $D(x_n, x_m) \to 0$, then $d(x_n, x_m) \to 0$ and $\frac{1}{d(x_n,G^c)}$ which is also a Cauchy sequence is bounded away from 0. Thus $x_n \to x \in X$ and $d(x, G^c) > 0$ making $x \in G$. This proves (G, D) is complete. If $A = \cap G_n$ we define

$$D(x,y) = d(x,y) + \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\frac{1}{d(x,G_j^c)} - \frac{1}{d(y,G_j^c)}|}{1 + |\frac{1}{d(x,G_j^c)} - \frac{1}{d(y,G_j^c)}|}$$

It is easy to check that on A D and d are equivalent, because if $x_n \to x$ and $x_n, x \in G_j$ for every j,

$$|\frac{1}{d(x_n, G_j^c)} - \frac{1}{d(x, G_j^c)}| \to 0$$

for every j, and it follows then that $D(x_n, x) \to 0$. If $D(x_n, x_m) \to 0$, just as before we conclude that $d(x_n, x) \to 0$ and $d(x, G_j^c) > 0$ for every j, which implies that $x \in \cap G_j = A$.