4.10 Brownian Motion on the Halfline.

It is not possible to construct the Brownian on the halfline $[0\,,\infty)$. Sooner or later it will hit 0 and then immeditely would turn negative as the following lemmas show.

Lemma 4.12. *For any $x \in R$, for the Brownian motion on R,*

$$P_x\left[\tau_0 < \infty\right] = 1$$

Proof. From the reflection principle for any $x > 0$,

$$P_x\left[\tau_0 \le t\right] = 2P_x\left[x(t) \le 0\right] \to 1 \quad \text{as} \quad t \to \infty.$$

□

Lemma 4.13. *(Blumenthal's 0-1 Law). If $P_{0,x}$ is any diffusion constructed as the unique martingale solution starting from x, and $A \in \mathcal{F}_{0+} = \cap s > 0 \mathcal{F}_s$, then $P_{0,x}(A) = 0$ or 1.*

Proof. Assume that $P_{0,x}(A) > 0$. Then $Q_A(\cdot)$ defined by

$$Q_A(E) = \frac{P_{0,x}(A \cap E)}{P_{0,x}(A)}$$

is easily seen to be again a martingale solution and so must coincide with $P_{0,x}$. Hence

$$P_{0,x}(A \cap E) = P_{0,x}(E)P_{0,x}(A)$$

In particular A and E are independent. Taking E to be A, we get $P(A) = 1$. □

Lemma 4.14. *For the Brownian Motion P_x starting from 0, for any $\delta > 0$,*

$$P_x\left[\omega : x(t) \ge 0 \quad \text{for} \quad 0 \le t \le \delta\right] = 0$$

Proof. For any $\delta > 0$

$$\begin{aligned} P_x\left[\cup_{\delta>0}\{\omega : x(t) \ge 0 \quad \text{for} \quad 0 \le t \le \delta\}\right] &= \lim_{\delta\to 0} P_x\left[\omega : x(t) \ge 0 \quad \text{for} \quad 0 \le t \le \delta\right] \\ &\le \lim_{\delta\to 0} P_x\left[\omega : x(\delta) \ge 0\right] \le \frac{1}{2} \end{aligned}$$

The set $A = \cup_{\delta>0}\{\omega : x(t) \ge 0 \quad \text{for} \quad 0 \le t \le \delta\}$ is in $\mathcal{F}_{0+}$ and by Lemma 4.13, $P_x(A) = 0$. □

We have to do something drastic to the Brownian Motion to keep it on the halfline. We want to characterize what we could do. We want to characterize all strong Markov families $\{P_x\}$ that have continuous paths, live on the half line and behave like a normal Brownian Motion away from 0. The last property is described by the following. For any smooth f that vanishes in a neighborhood of 0,

$$X_f(t) = f(x(t)) - f(x(0)) - \int_0^t \frac{1}{2} f''(x(s))ds$$

is a Martingale with respect to $(\Omega, \mathcal{F}_t, P_x)$. By approximation we can easily extend the property to functions f which are quadratic near ∞ while still vanishing near the origin. Hence such processes have two moments and in fact as many moments as we need. Any function with $f(0) = f'(0) = f''(0)$ can be approximated in the C^2 topology by functions that vanish in a neighborhood of 0. Constants are no problem. Therefore the martingale property is valid for all smooth functions f, that satisfy $f'(0) = f''(0) = 0$.

Lemma 4.15. *The function $x(t)$ is a submartingale with respect to any P_x and can be written as*

$$x(t) = A(t) + M(t) \tag{4.15}$$

where $M(t)$ is a martingale and $A(t)$ is a continuous nondecreasing function of t that increases only when $x(t)$ is at 0.

Proof. Approximate x by

$$f_\epsilon(x) = x - \epsilon \arctan \frac{x}{\epsilon}$$

Because

$$\frac{1}{2} f''_\epsilon(x) = g_\epsilon(x) = \frac{\epsilon x}{(\epsilon^2 + x^2)^2} \geq 0$$

$f_\epsilon(x(t))$ is a submartingale and in the limit so is $x(t)$. Existence of moments provides enough uniform integrability. Although a general theorem will tell us that a decomposition of the form (4.15) holds, we will do it by hand in this case. We obviously want to take

$$A(t) = \lim_{\epsilon \to 0} A_\epsilon(t) = \lim_{\epsilon \to 0} \int_0^t g_\epsilon(x(s))ds$$

Let us try to control

$$\begin{aligned}
&E\left[[A_\epsilon(t)]^2\right] \\
&= 2E\left[\int\int_{0 \leq t_1 \leq t_2 \leq t} g_\epsilon(x(t_1))g_\epsilon(x(t_2))dt_1 dt_2\right] \\
&= 2E\left[\int_{0 \leq t_1 \leq t} g_\epsilon(x(t_1))[f_\epsilon(x(t)) - f_\epsilon(x(t_1))]dt_1\right]
\end{aligned}$$

If we define

$$q_\epsilon(t) = \sup_{0 \leq s \leq t} \sup_x E^{P_x}\left[f_\epsilon(x(s)) - f_\epsilon(x(0))\right]$$

Then

$$E\left[\,[A_\epsilon(t)]^2\right] \le 2\,[q_\epsilon(t)]^2$$

Or more generally,

$$E\left[\,[A_\epsilon(t)]^k\right] \le k!\,[q_\epsilon(t)]^k$$

We next estimate $q_\epsilon(t)$.

$$\begin{aligned} q_\epsilon(t) &= \sup_{0\le s\le t}\sup_x E^{P_x}\left[(x(t)-\epsilon\arctan\frac{x(t)}{\epsilon}) - (x-\epsilon\arctan\frac{x}{\epsilon})\right] \\ &\le \sup_{0\le s\le t}\sup_x E^{P_x}\left[|x(t)-x|\right] \\ &\le \sup_{0\le s\le t}\sup_x \sqrt{E^{P_x}\left[|x(t)-x|^2\right]} \end{aligned}$$

We saw that $x(t)$ is a submartingale. By a similar argument one can show easily that $x^2(t)-t$ is a supermartingale. If we approximate x^2 by $h_\epsilon(x) = [x-\epsilon\arctan\frac{x}{\epsilon}]^2$

$$h''_\epsilon(x) \to \chi_{(0,\infty)}(x)$$

and is uniformly bounded. Therefore

$$x^2(t) - x^2(0) - \int_0^t \chi_{(0,\infty)}(x(s))ds$$

is a martingale. Therefore

$$\begin{aligned} E^{P_x}\left[|x(t)-x|^2\right] &= E^{P_x}\left[x^2(t) - 2x(t)x(0) + x^2(0)\right] \\ &\le E^{P_x}\left[x^2(0) + t - 2x^2(0) + x^2(0)\right] \\ &= t \end{aligned}$$

Providing us the estimate

$$q_\epsilon(t) \le k!t^{\frac{k}{2}}.$$

□

We will develop two methods for the construction of Brownian motions on the halfline with sticky boundary condition. The reflected Brownian motion exists as the family of distributions $\{P_x^0\}$ obtained from the Brownian motion measures $\{P_x\}$, by the map $P_x^0 = P_x\Phi^{-1}$ where Φ maps $C[[0\,,\infty);R]$ into $C[[0\,,\infty);R_+]$ by $\beta(\cdot)\to|\beta(\cdot)|$. Relative to any $(\Omega_+\,,\mathcal{F}_t\,,P_x^0)$ there is a local time $A(t)$ with the following properties:

1. $A(t)$ is nondecreasing and the support of the measure $dA(t)$ is contained in the set $\{t : x(t) = 0\}$.

2. For any smooth function f

$$f(x(t)) - f(x(0)) - \int_0^t \frac{1}{2}f''(x(s))ds - f'(0)A(t) \tag{4.16}$$

is a martingale relative to $(\Omega_+\,,\mathcal{F}_t\,,P_x^0)$.

3. The process $x(t)$ spends no time on the boundary 0, i.e. for any $x \in R_+$,

$$\int_0^t \chi_{\{0\}}(x(s))ds = 0 \quad \text{a.e.} \quad P_x^0 \tag{4.17}$$

We define a new increasing function

$$B(t) = \lambda^{-1} A(t) + t$$

where $\lambda > 0$ is a positive constant. $B(t)$ is a continuous strictly increasing function of t for any choice of $\lambda > 0$. For almost all ω the decomposition of B into

$$dB = \lambda^{-1}\, dA + dt$$

is its Lebesgue decomposition.

$$\text{support } dA = \{t : x(t) = 0\}$$

and, because of () we can take

$$\text{support } dt = \{t : x(t) > 0\}$$

We now conclude that the Radon-Nikodym derivatives are given by

$$\frac{dA}{dB} = \lambda\, \chi_{\{0\}}(x(s))$$

and

$$\frac{dt}{dB} = \chi_{(0,\infty)}(x(s))$$

We define τ_t as the solution of

$$B(\tau_t) = t$$

and define

$$y(t) = x(\tau_t).$$

Then

$$\begin{aligned}
f(y(t)) - f(y(0)) &- \int_0^t \frac{1}{2} f''(y(s))\chi_{(0,\infty)}(y(s))ds - \lambda f'(0)\int_0^t \chi_{\{0\}}(y(s))ds \\
&= f(x(\tau_t)) - f(y(0)) - \int_0^t \frac{1}{2} f''(x(\tau_s))\chi_{(0,\infty)}(x(\tau_s))ds \\
&\quad - \lambda f'(0)\int_0^t \chi_{\{0\}}(x(\tau_s))ds \\
&= f(x(\tau_t)) - f(x(0)) - \int_0^{\tau_t} \frac{1}{2} f''(x(\tau_{B(s)}))\chi_{(0,\infty)}(x(\tau_{B(s)}))dB(s) \\
&\quad - \lambda f'(0)\int_0^{\tau_t} \chi_{\{0\}}(x(\tau_{B(s)}))dB(s) \\
&\qquad \text{(by change of variables } \quad s \to B(s)\,) \\
&= f(x(\tau_t)) - f(x(0)) - \int_0^{\tau_t} \frac{1}{2} f''(x(s))\chi_{(0,\infty)}(x(s))dB(s) \\
&\quad - \lambda f'(0)\int_0^{\tau_t} \chi_{\{0\}}(x(s))dB(s) \\
&= f(x(\tau_t)) - f(x(0)) - \int_0^{\tau_t} \frac{1}{2} f''(x(s))ds - \lambda f'(0)\int_0^{\tau_t} dA(s) \\
&= f(x(\tau_t)) - f(x(0)) - \int_0^{\tau_t} \frac{1}{2} f''(x(s))ds - \lambda f'(0)A(\tau_t)
\end{aligned}$$

is a martingale with respect to $(\Omega\,,\mathcal{F}_{\tau_t}\,,P_x^0)$. Since the σ-field $\sigma\{y(s) : 0 \le s \le t\} \subset \mathcal{F}_{\tau_t}$ we conclude that the distributions $\{P_x^\lambda\}$ of $y(\cdot)$ have the property:

$$f(y(t)) - f(y(0)) - \int_0^t \frac{1}{2} f''(y(s))\chi_{(0,\infty)}(y(s))ds - \lambda f'(0)\int_0^t \chi_{\{0\}}(y(s))ds$$

are $(\Omega_+\,,\mathcal{F}_t\,,P_x^\lambda)$ martingales. Speeding up the clock at the boundary so that the local time at the boundary turns into real time converts the reflected case to the sticky case. Conversely if we stop the clock when the process is at the boundary, any sticky case will become the reflected case.

Let us cosider the sticky case and define the function

$$B(t) = \int_0^t \chi_{(0,\infty)}(x(s))ds.$$

We then define τ_t by

$$B(\tau_t) = t$$

and $y(\cdot)$ by

$$y(t) = x(\tau_t)$$

To begin we need a lemma.

Lemma 4.16. *Relative to any P_x^λ, the function $B(t)$ is almost surely strictly increasing in t. In other words, although the process sticks at the boundary it never spends a positive 'interval 'of time at the boundary.*

Proof. The proof amounts to showing that if we start at the boundary, then

$$P_0^\lambda\left[\inf\{t : x(t) > 0\} = 0\right] = 1$$

Let us define

$$\tau = \inf\{t : x(t) > 0\}$$

Although τ is not quite a stopping time, it almost is, in the sense that $\tau + \epsilon$ is a stopping time for every $\epsilon > 0$. By working with $\tau + \epsilon$ and letting ϵ go to 0 at the end the strong Markov property is seen to hold for τ. By Blumenthal's $0 - 1$ law,

$$P_x\left[\tau = 0\right] = 0 \quad \text{or} \quad 1$$

If it is 1 we are done. If it is 0, at the end of this time τ, the process is still at 0 but now 'knows' that it should get out. Clearly a violation of the strong Markov property. □

Now we return to our main goal. We know that

$$f(x(t)) - f(x(0)) - \int_0^t \frac{1}{2} f''(x(s))\chi_{(0,\infty)}(x(s))ds - \lambda f'(0)\int_0^t \chi_{\{0\}}(x(s))ds$$

is a martingale. with respect to $(\Omega\,, \mathcal{F}_t\,, P_x^\lambda)$. Therefore for f satisfying the boundary condition $f'(0) = 0$,

$$\begin{aligned} f(x(\tau_t)) - f(x(0)) &- \int_0^{\tau_t} \frac{1}{2} f''(x(s))\chi_{(0,\infty)}(x(s))ds \\ &= f(y(t)) - f(x(0)) - \int_0^t \frac{1}{2} f''(y(s))ds \end{aligned}$$

is a martingale and we are done.

Example 4.1. Let us try to calculate

$$p_\lambda(t) = P_0^\lambda\left[x(t) = 0\right]$$

We try to calculate

$$p_\lambda(x\,, t) = P_x^\lambda\left[x(t) = 0\right]$$

by solving the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}$$

with the boundary condition

$$\lambda u_x(0) = \frac{1}{2} u_{xx}(0)$$

and the initial condition

$$u(0\,,x) = \chi_{\{0\}}(x)$$

The Laplace transform

$$v(\sigma\,,x) = \int_0^\infty e^{-\sigma t} u(t\,,x)dt$$

solves

$$\sigma v - \frac{1}{2} v_{xx} = 0 \quad \text{for} \quad x > 0$$

with the boundary condition

$$\sigma v(0) - \lambda v_x(0) = 1$$

Clearly

$$v_\sigma(x) = a \exp[-\sqrt{2\sigma}\, x]$$

with

$$a\left[\sigma + \lambda\sqrt{2}\sigma\right] = 1$$

or

$$\alpha = a(\sigma\,,\lambda) = \left[\sigma + \lambda\sqrt{2\sigma}\right]^{-1}$$

Hence

$$\int_0^\infty p_\lambda(t) e^{-\sigma t} dt = \left[\sigma + \lambda\sqrt{2\sigma}\right]^{-1}$$

This can be explicitly inverted to yield

$$p_\lambda(t) = \int_0^\infty \sqrt{\frac{2}{\pi t}} e^{-\frac{x^2}{2t} - 2\lambda x} dx = \int_0^\infty \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2} - 2\lambda x\sqrt{t}} dx$$

Let P_0^0 be the reflected Brownian Motion starting from 0. The distribution of the local time process $A(t)$ can be found exactly.

Theorem 4.17. *The process $A(t)$ has the same distribution as that of the process*

$$M(t) = \sup_{0 \le s \le t} \beta(t)$$

of the maximum of a Brownian motion starting from 0.

Proof. By Tanaka formula

$$x(t) = |\beta(t)| = \int_0^t \text{ sign } (\beta(s)) d\beta(s) + A(t) = z(t) + A(t)$$

where $z(\cdot)$ is again a Browninan Motion process and $A(t)$ is the local time process. We will establish that

$$A(t) = -\inf_{0 \le s \le t} z(s) = \sup_{0 \le s \le t} [-z(s)]$$

In fact let f, g and A be arbitrary continuous functions with $f(t) \equiv g(t) + A(t)$, $f(0) = g(0) = A(0) = 0$, $f \geq 0$ and $A(\cdot)$, nondecreasing and increasing only when $f(t) = 0$, i.e. support of dA is contained in $\{t : f(t) = 0\}$. Then f and A are uniquely determined by g and

$$A(t) = \sup_{0 \leq s \leq t} [-g(s)] \tag{4.18}$$

It is easy to see that with the choice (4.18) for $A(t)$, and $f(t) = g(t) + A(t)$ we get $f(t) \geq 0$ as well as $\{\text{support of dA}\} \subset \{\text{t : f(t) = 0}\}$. We will now prove uniqueness. Let

$$f_i(t) = g(t) + A_i(t); \quad i = 1, 2$$

with

$$\{\mathrm{support\ of\ dA_i}\} \subset \{\mathrm{t : f_i(t) = 0}\} \quad \mathrm{i = 1, 2.} \tag{4.19}$$

We have

$$f_1(t) - f_2(t) = A_1(t) - A_2(t).$$

Since $A_1(t) - A_2(t)$ is a function of bounded variation, using (4.19)

$$\begin{aligned}
[A_1(t) - A_2(t)]^2 &= \int_0^t [f_1(s) - f_2(s)][dA_1(s) - dA_2(s)] \\
&= - \int_0^t f_1(s) dA_2(s) - \int_0^t f_2(s) dA_1(s) \\
&\leq 0
\end{aligned}$$

giving us uniqueness. □

In particular we have

$$P_0^0 \left[A(t) \geq \ell\right] = P_0 \left[\sup_{0 \leq s \leq t} \beta(s) \geq \ell\right] = \int_\ell^\infty \sqrt{\frac{2}{\pi t}} \exp[-\frac{x^2}{2\, t}] dx$$

4.11 Reflected Processes in Higher Dimensions.

We will quickly describe some multidimensional generalizations of reflected Brownian Motion. Let G be a smooth region in R^d and $a = \{a_{i,j}(x)\}$, $b = \{b_i(x)\}$, coefficients that are 'nice ', i.e. a is smooth and positive definite and b is smooth. We want to construct a solution and we need to describe what happens when the path reaches the boundary. We will deal exclusively with the the reflected case and just make some comments at the end regarding other possibilities. Reflection is a bad choice for the name, but in reality the process gets kicked in, in some direction pointing to the interior as soon as it reaches the boundary. So we have a direction $J(b)$ pointing to the interior at every point $b \in B = \partial G$. We want to show that given a, b, G and J, there is a unique family of solutions $\{P_x : x \in G \cup B\}$ on $\Omega = C[[0\,, \infty); G \cup B]$ with the following properties.

1. $P_x[x(0) = x] = 1$

2. $P_x\left[\int_0^t \chi_B(x(s))ds = 0\right] = 1$

3. For any smooth function f that satisfies $< J(b)\,,(\nabla f)(b) > \geq 0$ on B,

$$f(x(t)) - f(x(0)) - \int_0^t (\mathcal{L}f)(x(s))ds$$

is a submartingale with respect to $(\Omega\,,\mathcal{F}_t\,,P_x)$.

The question of existence is a question of nonexplosion as well. To avoid the problem of dealing with this issue let us assume that our domain G is bounded. Then the question is purely local. If we start from $x \in G$ we know what happens until we reach the boundary. We do not see it. P_x is just the same as the solution with no boundary until the exit time from G. We therefore need to construct local solutions when we start on or near the boundary. This is carried out in several steps.

Step 1. Make a change of coordinates so that a boundary point b becomes 0 and the boundary becomes $x_1 = 0$, a straightline near that point. This will reduce the problem to a half space. The direction J on $B = \{x : x_1 = 0\}$ can be described by $(1, J_2(x_2, \cdots, x_d), \cdots, J_d(x_2, \cdots, x_d))$.

Step 2. Now make another change of coordinates of a special type, $x_1 \to x_1$, $x_i \to x_i - x_1\, J_i(x_2\,, \cdots, x_d)$ for $2 \leq i \leq d$. The bounadry remains the same, but the new direction J is just $(1, 0, \cdots, 0)$, the inward normal.

Step 3. By a Girsanov formula which can be extended to this case we can assume that $b = 0$.

Step 4. In the current coordinate system $a_{1,1}(x)$ is a strictly positive function and we can do a random time change using τ_t defined by

$$\int_0^{\tau_t} a_{1,1}(x(s))ds = t$$

to reduce it to $a_{1,1} \equiv 1$. At this point if $f = f(x_1)$ is a function of x_1 only then

$$(\mathcal{L}f)(x) = \frac{1}{2}f''(x_1)$$

so that the process $x_1(t)$ is in fact the one dimensional reflected Brownian Motion.

Step 5. We can find a square root $\sigma(x)$ for $a(x)$ such that $a(x) = \sigma(s)\sigma^*(x)$ with $\sigma_{1,1}(x) \equiv 1$ and $\sigma_{1,j}(x) \equiv 0$ for $2 \leq j \leq d$. The stochastic differential equations for $x(t)$ now look like

$$dx_1(t) = d\beta(t) + A(t)$$

which is the decomposition of the reflected one dimensional Brownian motion and is already solved.

$$\begin{aligned}dx_j(t) &= \sigma_{j,1}(x_1(t), x_2(t), \cdots, x_d(t))dx_1(t)\\ &\quad + \sum_{2\le k\le d} \sigma_{j,k}(x_1(t), x_2(t), \cdots, x_d(t))dx_j(t)\end{aligned}$$

which can be solved by iteration for $x_2(\cdot), \cdots, x_d(\cdot)$ because the boundary has no effect on them directly.

Comments: We may try to stick to the boundary a little bit. This is dealt the same way as in one dimension. We can obtain it by random time change from the reflected case using the local time on the boundary. The holding rate ρ can now be a function $\rho(b)$ defined on B. The local time $A(t)$ in the reflected case can be used to construct the time change

$$\int_0^{\tau_t} \lambda(x(s))dA(s) + \tau_t = t$$

where $\lambda(b) = [\rho(b)]^{-1}$. Finally a new phenomenon that can happen is that the path might diffuse on the boundary which amounts to the kick having a random tangential component. Imagine in the case of a halfspace, being kicked form the boundary point $(0, y)$, to the interior point $(\delta\,, y + \delta J(y) + \sqrt{\delta}\xi)$ where ξ is a gaussian random vector with mean 0 and covariance matrix $D(y)$. The boundary condition then becomes

$$(\mathcal{B}f)(b) = \frac{\partial f}{\partial x_1} + \sum_{j=2}^{d} J_j(y)\frac{\partial f}{\partial x_j} + \frac{1}{2}\sum_{i,j=1}^{d} D_{i,j}(y)\frac{\partial^2 f}{\partial x_i \partial x_j} = 0$$

at $b = (0, y) \in B$. Here y refers to the cordinates x_2 through x_d. Of course this can happen in the sticky situation as well and the boundary condition then is

$$(\mathcal{L}f)(b) = \rho(b)(\mathcal{B}f)(b)$$