8 BMO

The space of functions of Bounded Mean Oscillation (BMO) plays an important role in Harmonic Analysis.

A function f, in $L_1(loc)$ in R^d is said to be a **BMO** function if

$$\sup_{x,r} \inf_a \frac{1}{|B_{x,r}|} \int_{y \in B_{x,r}} |f(y) - a| dy = \|u\|_{BMO} < \infty \tag{8.1}$$

where $B_{x,r}$ is the ball of radius r centered at x, and $|B_{x,r}|$ is its volume.
Remark. The infimum over a can be replaced by the choice of

$$a = \bar{a} = \frac{1}{|B_{x,r}|} \int_{y \in B_{x,r}} f(y) dy$$

giving us an equivalent definition. We note that for any a,

$$|a - \bar{a}| \leq \frac{1}{|B_{x,r}|} \int_{y \in B_{x,r}} |f(y) - a| dy$$

and therefore if a^* is the optimal a,

$$|\bar{a} - a^*| \leq \|f\|_{BMO}$$

Remark. Any bounded function is in the class BMO and $\|f\|_{BMO} \leq \|f\|_\infty$.

Theorem 8.1 (John-Nirenberg). *Let f be a BMO function on a cube Q of volume $|Q| = 1$ satisfying $\int_Q f(x)dx = 0$ and $\|f\|_{BMO} \leq 1$. Then there are finite positive constants c_1, c_2, independent of f, such that, for any $\ell > 0$*

$$|\{x : |f(x)| \geq \ell\}| \leq c_1 \exp[-\frac{\ell}{c_2}] \tag{8.2}$$

Proof. Let us define

$$F(\ell) = \sup_f |\{x : |f(x)| \geq \ell\}$$

where the supremum is taken over all functions with $\|f\|_{BMO} \leq 1$ and $\int_Q f(x)dx = 0$. Since $\int_Q f(x)dx = 0$ implies that $\|f\|_1 \leq \|f\|_{BMO} \leq 1$, $F(\ell) \leq \frac{1}{\ell}$. Let us subdivide the cube into 2^d subcubes with sides one half the original cube. We pick a number $a > 1$ and keep the cubes Q_i with $\frac{1}{|Q_i|} \int_{Q_i} |f(x)| dx \geq a$. We subdivide again those with $\frac{1}{|Q_i|} \int_{Q_i} |f(x)| dx < a$ and keep going. In this manner we get an atmost countable collection of disjoint cubes that we enumerate as $\{Q_j\}$, that have the following properties:

1. $\frac{1}{|Q_j|}\int_{Q_j}|f(x)|dx \geq a$.

2. Each Q_j is contained in a bigger cube Q'_j with sides double the size of the sides of Q_j and $\frac{1}{|Q'_j|}\int_{Q'_j}|f(x)|dx < a$.

3. By the Lebesgue theorem $|f(x)| \leq a$ on $Q \cap (\cup_j Q_j)^c$.

If we denote by $a_j = \frac{1}{|Q_j|}\int_{Q_j} f(x)dx$, we have

$$|a_j| \leq \frac{1}{|Q_j|}\int_{Q_j}|f(x)|dx \leq \frac{2^d}{|Q'_j|}\int_{Q'_j}|f(x)|dx \leq 2^d a$$

by property 2). On the other hand $f - a_j$ has mean 0 on Q_j and BMO norm at most 1. Therefore (scaling up the cube to standard size)

$$\begin{aligned}|Q_j \cap \{x : |f(x)| \geq 2^d a + \ell\}| &\leq |Q_j \cap \{x : |f(x) - a_j| \geq \ell\}| \\ &\leq |Q_j| F(\ell)\end{aligned}$$

Summing over j, because of property 3),

$$|\{x : |f(x)| \geq 2^d a + \ell\}| \leq F(\ell)\sum_j |Q_j|$$

On the other hand property 1) implies that $\sum_j |Q_j| \leq \frac{1}{a}$ giving us

$$F(2^d a + \ell) \leq \frac{1}{a}F(\ell)$$

which is enough to prove the theorem. □

Corollary 8.1. *For any $p > 1$ there is a constant $C_{d,p}$ depending only on the dimension d and p such that*

$$\sup_Q \frac{1}{|Q|}\int_Q |f(x) - \frac{1}{|Q|}\int_Q f(x)dx|^p dx \leq C_{d,p}\|f\|_{BMO}^p$$

The importance of BMO, lies partly in the fact that it is dual to $\mathcal{H}_1$.

Theorem 8.2. *There are constants $0 < c \leq C < \infty$ such that*

$$c\|f\|_{BMO} \leq \sup_{g:\|g\|_{\mathcal{H}_1}\leq 1} |\int f(x)g(x)dx| \leq C\|f\|_{BMO} \tag{8.3}$$

and every bounded linear functional on $\mathcal{H}_1$ is of the above type.

The proof of the theorem depends on some lemmas.

Lemma 8.1. *The Riesz transforms R_i map $L_\infty \to BMO$ boundedly. In fact convolution by any kernel of the form $K(x) = \frac{\Omega(x)}{|x|^d}$ where $\Omega(x)$ is homogeneous of degree zero, has mean 0 on S^{d-1} and satisfies a Hölder condition on S^{d-1} maps $L_\infty \to$BMO boundedly.*

Proof. Let us suppose that Q is the unit cube centered around the origin and denote by $2Q$ the doubled cube. We write $f = f_1 + f_2$ where $f_1 = f\mathbf{1}_{2Q}$ and $f_2 = f - f_1 = f\mathbf{1}_{(2Q)^c}$.

$$g(x) = g_1(x) + g_2(x)$$

where

$$g_i(x) = \int_{R^d} K(x-y) f_i(y) dy$$

$$\int_Q |g_1(x)| dx \le \|g_1\|_2 \le \sup_\xi |\widehat{K}(\xi)| \|f_1\|_2 \le 2^{\frac{d}{2}} \sup_\xi |\widehat{K}(\xi)| \|f\|_\infty$$

On the other hand with $a_Q = \int_Q K(-y) f_2(y) dy$

$$\begin{aligned}
&\int_Q |g_2(x) - a_Q| dx \\
&\qquad \le \int_Q dx \int_{R^d} |K(x-y) - K(-y)| f_2(y) dy \\
&\qquad \le \|f\|_\infty \int\int_{\substack{z\in Q \\ y\notin 2Q}} |K(x-y) - K(-y)| dx dy \\
&\qquad \le \|f\|_\infty \sup_x \int_{|y|\ge 2|x|} |K(x-y) - K(-y)| dy \\
&\qquad \le B\|f\|_\infty
\end{aligned}$$

The proof for arbitrary cube is just a matter of translation and scaling. The Hölder continuity is used to prove the boundedness of $\widehat{K}(\xi)$. □

Lemma 8.2. *Any bounded linear function Λ on $\mathcal{H}_1$ is given by*

$$\Lambda(f) = \sum_{i=0}^{d} \int (R_i f)(x) g_i(x) dx = -\int f(x) \sum_{i=0}^{d} (R_i g_i)(x) dx$$

where $R_0 = \mathcal{I}$ and R_i for $1 \le i \le d$ are the Riesz transforms.

Proof. The space $\mathcal{H}_1$ is a closed subspace of the direct sum $\oplus L_1(R^d)$ of $d+1$ copies of $L_1(R^d)$. Hahn-Banach theorem allows us to extend Λ boundedly to $\oplus L_1(R^d)$ and the Riesz representation theorem gives us $\{g_i\}$. Finally $g_0 + \sum_{i=1}^d R_i g_i$ is in BMO. □

Lemma 8.3. *If $g \in$BMO then*

$$\int_{R^d} \frac{|g(y)|}{1+|y|^{d+1}} dy < \infty \tag{8.4}$$

and

$$G(t,x) = \int g(y) p(t, x-y) dy$$

exists where $p(\cdot\,,\cdot)$ is the Poisson kernel for the half space $t > 0$. Moreover $g(t,x)$ satisfies

$$\sup_x \int_{\substack{|y-x|<h \\ 0<t<h}} t|\nabla G(t,y)|^2 dt dy \leq A\|g\|_{BMO}^2 h^d \tag{8.5}$$

for some constant independent of g. Here ∇G is the full gradient in t and x.

Proof. First let us estimate $\int_{R^d} \frac{|g(x)|}{1+|x|^{d+1}} dx$. If we denote by Q_n the cube of side 2^n around the origin

$$\begin{aligned}
\int_{R^d} \frac{|g(x)|}{1+|x|^{d+1}} dx &\leq \int_{Q_0} \frac{|g(x)|}{1+|x|^{d+1}} dx + \sum_n \int_{Q_{n+1}\cap Q_n^c} \frac{|g(x)|}{1+|x|^{d+1}} dx \\
&\leq \int_{Q_0} |g(x)| dx + \sum_n \frac{1}{2^{n(d+1)}} \int_{Q_{n+1}} |g(x)| dx \\
&\leq \int_{Q_0} |g(x)| dx + \sum_n \frac{1}{2^{n(d+1)}} \int_{Q_{n+1}} |g(x) - a_{n+1}| dx \\
&\quad + \sum_n \frac{|a_{n+1}|}{2^n} \\
&\leq \int_{Q_0} |g(x)| dx + \|g\|_{BMO} \sum_n \frac{2^{(n+1)d}}{2^{n(d+1)}} + \sum_n \frac{|a_{n+1}|}{2^n}
\end{aligned}$$

where

$$a_{n+1} \leq \frac{1}{2^{(n+1)d}} \int_{Q_{n+1}} g(x)dx$$

Moreover

$$|a_{2Q} - a_Q| = \frac{1}{|Q|} \int_Q |g(x) - a_{2Q}|dx \leq 2^d \|g\|_{BMO}$$

and this provides a bound of the form

$$|a_{Q_n}| \leq Cn\|g\|_{BMO} + \int_{Q_0} |g(x)|dx$$

establishing (8.4). We now turn to proving (8.5). Again because of the homogeneity under translations and rescaling, we can assume that $x = 0$ and $h = 1$. So we only need to control

$$\int_{\substack{|y|<1 \\ 0<t<1}} t|\nabla G(t,y)|^2 dtdy \leq A\|g\|_{BMO}^2$$

We denote by Q_4 the cube $|x| \leq 2$ and write g as

$$g = a_{Q_4} + (g_1 - a_{Q_4}) + g_2$$

where $g_1 = g\mathbf{1}_{Q_4}$, $g_2 = g - g_1 = g\mathbf{1}_{Q_4^c}$. Since constants do not contribute to (8.5), we can assume that $a_{Q_4} = 0$, and therefore the integral $\int_{Q_4} |g(x)|dx$ can be estimated in terms of $\|g\|_{BMO}$. An easy calculation, writing $G = G_1 + G_2$ yields

$$|\nabla G_2(t,y)| \leq \int_{Q_4^c} \frac{|g(x)|}{1 + |x|^{d+1}} dx \leq A\|g\|_{BMO}$$

As for the G_1 contribution in terms of the Fourier transform we can control it by

$$\int_0^\infty \int_{R^d} t|\nabla G|^2 dtdy = \int_0^\infty \int_{R^d} t|\xi|^2 e^{-2t|\xi|} |\widehat{g}_1(\xi)|^2 d\xi dt = \int_{R^d} |\widehat{g}_1(\xi)|^2 d\xi$$

which is controlled by $\|g\|_{BMO}$ because of the John-Nirenberg theorem. $\square$

Lemma 8.4. *Any function g whose Poisson intgeral G satisfies (8.5) defines a bounded linear functional on $\mathcal{H}_1$.*

Proof. The idea of the proof is to write

$$\begin{aligned} 2\int_0^\infty \int_{R^d} t\nabla G(t,x)\nabla F(t,x)dtdx &= 4\int_0^\infty \int_{R^d} te^{-2t|\xi|}|\xi|^2 \widehat{f}(\xi)\bar{\widehat{g}}(\xi)d\xi dt \\ &= \int_{R^d} \widehat{f}(\xi)\bar{\widehat{g}}(\xi)d\xi \\ &= \int_{R^d} f(x)g(x)dx \end{aligned}$$

and concentrate on

$$\int_0^\infty \int_{R^d} t|\nabla_x G(t,x)||\nabla_x F(t,x)|dtdx$$

We need the auxiliary function

$$(S_h u)(x) = [\int\int_{|x-y|<t<h} t^{1-d}|\nabla u|^2 dydt]^{\frac{1}{2}}$$

Clearly $(S_h u)(x)$ is increasing in h and we show in the next lemma that

$$\|S_\infty F\|_1 \leq C\|f\|_{\mathcal{H}_1}$$

Let us assume it and complete the proof. Define

$$h(x) = \sup\{h : (S_h F)(x) \leq MC\}$$

then

$$(S_{h(x)}F)(x) \leq MC$$

In addition it follows from (8.5) that

$$\sup_{y,h} \int_{|y-x|\leq h} |(S_h F)(x)|^2 dx \leq Ch^d$$

Now $h(x) < h$ means $(S_h F)(x) > MC$ and therefore

$$|\{x : |x-y| < h, h(x) < h\}| \leq \frac{Ch^d}{M^2}$$

By the proper choice of M, we can be sure that

$$|\{x : |x-y| < h, h(x) \geq h\}| \geq ch^d$$

Now we complete the proof.

$$\begin{aligned}
&\int_0^\infty \int_{R^d} t|\nabla_x G(t,x)||\nabla_x F(t,x)|dtdx \\
&\qquad \leq C\int_0^\infty \int_{R^d}\int_{|y-x|<t\leq h(y)} t^{1-d}|\nabla_x G(t,x)||\nabla_x F(t,x)|dtdxdy \\
&\qquad \leq \int_{R^d} dy(\int_0^\infty \int_{|y-x|<t\leq h(y)} t^{1-d}|\nabla_x G(t,x)|^2 dxdt)^{\frac{1}{2}} \\
&\qquad\qquad \times (\int_0^\infty \int_{|y-x|<t\leq h(y)} t^{1-d}|\nabla_x F(t,x)|^2 dxdt)^{\frac{1}{2}} \\
&\qquad \leq M\int_{R^d} (S_\infty F)(y)dy \leq M\|f\|_{\mathcal{H}_1}
\end{aligned}$$

□

Lemma 8.5. *If $f \in \mathcal{H}_1$ then $|(S_\infty F)(x)|_1 \leq C\|f\|_{\mathcal{H}_1}$.*

Proof. This is done in two steps.

Step 1. We control the nontangential maximal function

$$U^*(x) = \sup_{y,t:|x-y|\leq kt} |U(t,y)|$$

by

$$\|U^*\|_1 \leq C_k\|u\|_{\mathcal{H}_1}$$

If $U_0(x) \in \mathcal{H}_1$ then U_0 and its n Riesz transforms $U_1, \dots, U_n$ can be recognized as the full gradient of a Harmonic function W on R_+^{n+1}. Then $V = (U_0^2 + \cdots + U_n^2)^{\frac{p}{2}}$ can be verified to be subharmonic provided $p > \frac{n-1}{n-2}$. This depends on the calculation

$$\begin{aligned}
\Delta V &= \frac{p}{2}\frac{p-2}{2}V^{\frac{p}{2}-2}\|\nabla V\|^2 + \frac{p}{2}V^{\frac{p}{2}-1}\Delta V \\
&= pV^{\frac{p}{2}-2}\left[(p-2)\|\sum U_j\nabla U_j\|^2 + V\sum_j \|\nabla U_j\|^2\right] \\
&\geq 0
\end{aligned}$$

provided either $p \geq 2$, or if $0 < p < 2$,

$$\|H\xi\|^2 \leq \frac{1}{2-p} \text{ Tr } (H^*H)\|\xi\|^2 \tag{8.6}$$

where H is the Hessian of W with trace 0 and $\xi = (U_0, \ldots, U_n)$. Then if $\{\lambda_j\}$ are the $n+1$ eigenvalues of H, and λ_0 is the one with largest modulus, the remaining ones have an average of $-\frac{\lambda_0}{n}$ and therefore

$$\text{Tr } (H^*H) = \sum \lambda_j^2 \geq (1 + \frac{1}{n})\lambda_0^2$$

This means that for equation (8.6) to hold we only need $\frac{n}{n+1} \leq \frac{1}{2-p}$ or $p \geq \frac{n-1}{n+1}$. In any case there is a choice of $p = p_n < 1$ that is allowed.

Now consider the subharmonic function V. If we denote by $h(t,x)$ the Poisson integral of the boundary values of $h(0,x) = V(0,x)$,

$$V(t,x) \leq h(t,x)$$

and we have

$$U^*(x) = \sup_{(y,t):\|x-y\| \leq kt} U(t,y) \leq \sup_{(y,t):\|x-y\| \leq kt} V[(t,y)]^{\frac{1}{p}} \leq \sup_{(y,t):\|x-y\| \leq kt} h[(t,y)]^{\frac{1}{p}}$$

By maximal inequality, valid because $\frac{1}{p} > 1$,

$$\|U^*\|_1 \leq \|h^*\|^p_{\frac{1}{p}} \leq C_{k,p}\|h(0,x)\|^p_{\frac{1}{p}} = C_{k,p}\|V(0,x)\|^p_{\frac{1}{p}} \leq C_k\|U\|_{\mathcal{H}_1}$$

Step 2. It is now left to control $\|(S_\infty U)(x)\|_1 \leq C\|U^*\|_1$. We use the room between the regions $|x-y| \leq t$ in the defintion of S and the larger regions $|x-y| \leq kt$ used in the definition of U^* to control S through U. Let us pick $k = 4$. Let $\alpha > 0$ be a number. Consider the set $E = \{x : |U^*(x)| \leq \alpha$ and $B = E^c = \{x : |U^*(x)| > \alpha\}$. We denote by G the union $G = \cup_{x \in E}\{(t,y) : |x-y| \leq t\}$. We want to estimate

$$\begin{aligned}
\int_E |S_\infty U|^2(x)dx &= \int\int\int_{\substack{x \in E \\ |x-y| \leq t}} t^{1-d}|\nabla U|^2(t,y)dxdtdy \\
&\leq C\int_G t|\nabla U|^2(t,y)dtdy \\
&\leq C\int_G t(\Delta U^2)(t,y)dtdy \\
&\leq C\int_{\partial G}[|t\frac{\partial U^2}{\partial n}(t,y)| + |U^2(t,y)\frac{\partial t}{\partial n}(t,y)|]d\sigma
\end{aligned}$$

by Greens's theorem. We have cheated a bit. We have assumed some smoothness on ∂G. We have assumed decay at ∞ so there are no contributions from ∞. We can assume that we have initially $U(0,x) \in L_2$ so the decay is valid. We can approximate G from inside by regions G_ϵ with smooth boundary. The boundary consists of two parts. $B_1 = \{t = 0, x \in E\}$ and $B_2 = \{x \in E^c, t = \phi(x)\}$. Moreover $|\nabla\phi| \leq 1$. We will show below that $t|\nabla U(t,y)| \leq C\alpha$ in G. On B_1 one can show that $t|U||\nabla U| \to 0$ and $U^2 \frac{\partial t}{\partial n} \to U^2$. Moreover $d\sigma \simeq dx$. The contribution from B_1 is therefore bounded by $\int_E |U(0,x)|^2 dx \leq \int_E |U^*(0,x)|^2 dx$. On the other hand on B_2 since it is still true that $d\sigma = dx$, using the bound $t|\nabla U| \leq C\alpha$, $|\frac{\partial t}{\partial n}| \leq 1$, we see that the contribution is bounded by $C\alpha^2|E^c|$. Putting the pieces together we get

$$\begin{aligned}\int_E |S_\infty U|^2(x)dx &\leq C\alpha^2 T_{U^*}(\alpha) + C\int_E |U^*|^2(x)dx \\ &\leq C\alpha^2 T_{U^*}(\alpha) + C\int_0^\alpha zT_{U^*}(z)dz\end{aligned}$$

where $T_{U^*}(z) = \text{mes}\{x : |U^*(x)| > z\}$. Finally since $\text{mes}(E^c) = T_{U^*}(\alpha)$

$$\text{mes}\{x : |S_\infty U(x)| > \alpha\} \leq CT_{U^*}(\alpha) + \frac{C}{\alpha^2}\int_0^\alpha zT_{U^*}(z)dz$$

Integrating with respect to α we obtain

$$\|S_\infty U\|_1 \leq C\|U^*\|_1$$

Step 3. To get the bound $t|\nabla U| \leq C\alpha$ in G, we note that any $(t,x) \in G$ has a ball around it of radius t contained in the set $\cup_{x\in E}\{y : |x-y| \leq 4t\}$ where $|U| \leq \alpha$ and by standard estimates, if a Harmonic function is bounded by α in a ball of radius t then its gradient at the center is bounded by $\frac{C\alpha}{t}$. □