We now look at the nonlinearly perturbed version

$$dx(t) = Ax(t)dt + F(x(t))dt + Bd\beta(t)$$

where $F : \mathcal{H} \to \mathcal{H}$ is a Lipschitz map, i.e.

$$\|F(x) - F(y)\| \leq C\|x - y\|$$

A mild solution of the equation is one that satisfies which is almost surely in $C[[0, T], \mathcal{H}]$ and satisfies

$$X(t) = T(t)x + \int_0^t T(t-s)F(X(s))ds + \int_0^t T(t-s)Bd\beta(s)$$

Under the conditions

$$\int_0^t TrT(s)BB^*T^*(s)ds < \infty$$

which makes

$$w(t) = \int_0^t T(t-s)Bd\beta(s)$$

well defined, the integral equation has a unique solution $X(t, x)$. For fixed x and $w(t)$ consider the map $U : C[[0, T]; \mathcal{H}] \to C[[0, T]; \mathcal{H}]$ defined by

$$U(X)(t) = T(t)x + \int_0^t T(t-s)F(X(s))ds + \int_0^t T(t-s)Bd\beta(s)$$

Then

$$U(X)(t) - U(Y)(t) = \int_0^t T(t-s)[F(X(s)) - F(Y(s))]ds$$

and

$$\sup_{0\leq t\leq T} \|U(X)(t) - U(Y)(t)\| \leq TMe^{\omega T}C \sup_{0\leq t\leq T} \|X(t) - Y(t)\|$$

For T small enough this is a contraction and has a fixed point. Since the estimate on T is uniform in x, and $w(t)$, we can iterate and get global existence and uniqueness.

Dependence on initial condition. It is easy to see that if F is smooth then $X(t, x)$ is smooth in x. If we compute the derivative D_h in some direction $h \in \mathcal{H}$, then $Y(t) = Y(t, x, h) = D_h X(t, x)$ satisfies

$$Y(t) = T(t)h + \int_0^t T(t-s)(DF)(X(s, x)) \cdot Y(s)dt$$

If $\|DF\|$ is uniformly bounded as a linear map from $\mathcal{H} \to \mathcal{H}$, then $Y(t, x, h)$ is a linear map $Z(t, x)h$ and $Z(t, x) = DX(t, x)$, has a uniform bound $\sup_x \|Z(t, x)\| \leq k(t)$.

Possible Approximations. One can approximate A by $A_k = k[(I - k^{-1}A)^{-1} - I]$ as in Hille-Yosida theory. Also one can take only a finite number of Brownian motions and use

$$\sum_{j=1}^{k} e_j \beta_j(t)$$

where $e_1, \ldots, e_k$ are the first k elements of an ON basis in $\mathcal{K}$. These approximations are useful in generalizing the obvious relations from finite to infinite dimensions.

Clearly the process is Feller. one checks

$$\|X(t,x) - X(t,y)\| \le k(t)\|x - y\|$$

Strong Feller Property. Let P_t be the semigroup

$$(P_t\phi)(x) = E[\phi(X(t,x))]$$

Then $(P_{t-s}\phi)(X(s,x)) = E[\phi(X(t,x))|X(s,x)]$ is a martingale.

$$\phi(X(t,x)) = (P_t\phi)(x) + \int_0^t \langle (DP_{t-s}\phi)(X(s,x)),\, Bd\beta(s)\rangle$$

Multiply both sides by $\int_0^t < B^{-1}D_hX(s,x), d\beta(s) >$ and take expectations.

$$\begin{aligned}
&E\left[\phi(X(t,x))\int_0^t < B^{-1}D_hX(s,x), d\beta(s) > \right] \\
&\quad = E\left[\int_0^t \langle (DP_{t-s}\phi)(X(s,x)),\, Bd\beta(s)\rangle \int_0^t < B^{-1}D_hX(s,x), d\beta(s) > \right] \\
&\quad = E\left[\int_0^t < B^*DP_{t-s}\phi)(X(s,x)), B^{-1}D_hX(s,x) > ds\right] \\
&\quad = E\left[\int_0^t < DP_{t-s}\phi)(X(s,x)), D_hX(s,x) > ds\right] \\
&\quad = E\left[\int_0^t < DP_{t-s}\phi)(X(s,x)), D_hX(s,x) > ds\right] \\
&\quad = t(D_hP_t\phi)(x)
\end{aligned}$$

Therefore

$$(D_hP_t\phi)(x) = \frac{1}{t}E\left[\phi(X(t,x))\int_0^t < B^{-1}D_hX(s,x), d\beta(s) > \right]$$

If B has a bounded inverse then $P_t\phi$ is Lipschitz for $t > 0$, provided ϕ is bounded measurable. The bound depends only on the Lipschitz norm of F. We conclude the strong Feller property if B is invertible and F is Lipschitz. In fact we have a gradient bound

$$\|DP_t\phi\|_\infty \le \frac{C\|\phi\|_\infty}{\sqrt{t}}$$

Irreducibility. Given any $x_0, x_1 \in \mathcal{H}$, $\tau > 0$ and $\epsilon > 0$ we will show that there is a control $u(s) \in L_2[[0,\tau];\mathcal{K}]$ such that the solution of

$$x(t) = T(t)x_0 + \int_0^t T(t-s)F(x(s))ds + \int_0^t T(t-s)Bu(s)ds$$

comes with in ϵ of x_1 at time τ. Let us pick $z_0, z_1 \in \mathcal{D}(A)$ such that $\|x_i - z_i\| \leq \delta$ for $i = 0, 1$. Consider the path $z(t)$ joining z_0 and z_1 defined by

$$z(t) = \frac{T-t}{T} z_0 + \frac{t}{T} z_1$$

If we can pick $u(t)$ such that

$$z'(t) = Az(t) + F(z(t)) + Bu(t)$$

the solution starting from z_0 will end up at z_1. By stability the solution starting from x_0 will be close to z_1 as well. Since z_1 is close to x_1 this is enough. However range of B is only dense. So we can only find $u(t)$ such that

$$z'(t) = Az(t) + F(z(t)) + Bu(t) + \delta(t)$$

with $\|\delta(t)\| \leq \delta$. Now if $v(t)$ solves

$$v'(t) = Av(t) + F(v(t)) + Bu(t)$$

With $w(t) = z(t) - v(t)$

$$w'(t) = Aw(t) + [F(z(t)) - F(w(t))] + \delta(t)$$

Or

$$w(t) = T(t)w(0) + \int_0^t T(t-s)[F(z(s)) - F(v(s))]ds + \int_0^t T(t-s)\delta(s)ds$$

It is now easy to deduce from the Lipschitz condition on F that $w(T)$ is small if $w(0)$ and $\delta(s)$ are small.

Such approximate controllability and the strong Feller property imply irreducibility and the uniqueness of the invariant measure if it exists. For instance if A is a Borel set and $q(t_0, x_0, A) > 0$ for some $t_0 > 0$ and $x_0 \in \mathcal{H}$, then $q(t_0, y, A) \geq \delta > 0$ for y in a neighborhood N of x. The approximate controllability implies that $q(t, y, U) > 0$ for every y and open set U. By Chapman-Kolmogorov equations $q(t, x, A) > 0$ for all $x \in \mathcal{H}$ and $t > t_0$. This implies the uniqueness of the invariant measure if it exists.

A simple sufficient condition for the existence of an invariant distribution is the condition

$$< F(x) - F(y), x - y > \leq c\|x - y\|^2$$

with c satisfying $\omega + c = \omega_1 < 0$, where

$$\|T(t)x\| \leq Ce^{\omega t}\|x\|$$

Sketch of Proof: The system is highly contractive. If we denote by $x(t), y(t)$, two solutions with initial values x and y respectively, then

$$x(t) - y(t) = T(t)[x - y] + \int_0^t T(t-s)[F(x(s)) - F(y(s))]ds$$

and

$$\begin{aligned}
&\frac{d}{dt}\|x(t) - y(t)\|^2 \\
&\quad = 2 < x(t) - y(t), F(x(t)) - F(y(t)) > +2 < A(x(t) - y(t)), (x(t) - y(t)) > \\
&\quad \leq 2\omega_1 \|x(t) - y(t)\|^2
\end{aligned}$$

providing an exponential decay for $\|x(t) - y(t)\|$. In addition we can also have an estimate for

$$E[\|x(1) - x\|^2] \leq C(x)$$

This means that if we solve with $x(-n-1) = x$ the solution at time $-n$ differs from x by $C(x)$. By exponential decay the two solution differ by $e^{\omega_1 n}$ at time 0. So the limit of $x(0)$ with $x(-n) = x$ exists. The noise has to be consistent. The limit then is a random variable $x(0$ and its distribution is the invariant measure.

One can verify that the invariant density is absolutely continuous with respect to the Gaussian when $B = I$.