The subject of stochastic partial differential deals with the study of solutions of partial differential equations perturbed by noise. Just like the study of ordinary differential equations perturbed by noise leads to the study of Stochastic Differential Equations or Diffusion processes in R^n, SPDE in a sense is the study of diffusions in infinite dimensional spaces. However on R^n we have the theory of elliptic and parabolic partial differential equations with its connections to diffusions that is of great help. This is lacking in infinite dimensions.

The simplest examples are linear SPDEs. In finite dimensions a simple linear SDE is of the form

$$dx(t) = Ax(t)dt + Bd\beta(t)$$

where $x(t) \in R^n$, $A : R^n \to R^n$ and $B : R^d \to R^n$ are linear maps with $\beta(t)$ being the standard Brownian motion in R^d. This SPDE can be solved explicitly by variation of parameters. If we denote the solution of the linear ODE

$$dy(t) = Ay(t); y(0) = y$$

by

$$y(t) = T(t)y = e^{tA}y$$

then the solution of the SDE is given by

$$x(t) = T(t)x(0) + \int_0^t T(t-s)Bd\beta(s)$$

or with indices thrown in

$$x_i(t) = x_i(0) + \sum_{j=1}^{n}\sum_{k=1}^{d}\int_0^t T_{i,j}(t-s)B_{j,k}d\beta_k(t)$$

Formally this can be generalized to infinite dimensions. One can replace R^n and R^d by two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. The canonical Brownian motion on R^d is repaced by the canonical Brownian motion on $\mathcal{K}$ which is defined as a "process" with independent increments

$$E[< z, \beta(t) - \beta(s) >< z', \beta(t) - \beta(s) >] =< z, z' > |t - s|$$

The only problem with the canonical Brownian motion is that it does not exist. Since $< e_i, \beta(t) - \beta(s) >$ are independent Gaussians with mean 0 and variance $|t - s|$, for $\{e_i\}$ that are orthonormal, the series

$$\sum_i | < e_i, \beta(t) - \beta(s) > |^2$$

is almost surely divergent. So the increments do not live on $\mathcal{K}$ but have to be supported on a larger space. Think of the norm on $\mathcal{K}$ as $\int_0^1 |f'(t)|^2dt$ for the standard BM that only lives on continuous function and not on functions with a square integrable derivative. But let us proceed and see what we need.

If we are to think of $x(t)$ as a process that lives on $\mathcal{H}$ we should be able to control $E[\|x(t)\|^2]$. This can be done. Assume that A generates a semigroup of bounded operators $\|T_t\| \leq Ce^{ct}$, then we only need to control $E[\|w(t)\|^2]$ where

$$w(t) = \int_0^t T(t-s)Bd\beta(s)$$

One can calculate the covariance of $w(t)$ as

$$E[< z, w(t) >< z', w(t) >] =< z, C(t)z' >$$

where

$$C(t) = \int_0^t T(t-s)BB^*T^*(t-s)ds = \int_0^t T(s)BB^*T^*(s)ds$$

$E[\|w(t)\|^2]$ can now be calculated as

$$TrC(t) = Tr \int_0^t T(t-s)BB^*T^*(t-s)ds = \int_0^t Tr[T(s)BB^*T^*(s)]ds$$

The natural assumption is to assume that B and $T(t)$ are together sufficiently compact that $T(s)B$ is Hilbert-Schmidt for $s > 0$ and the trace $Tr[T(s)BB^*T^*(s)]$ is integrable on $[0, t]$. If that be the case it is not hard to see that $x(t)$ as defined in

$$x(t) = T(t)x(0) + \int_0^t T(t-s)Bd\beta(s)$$

makes sense and defines a Gaussian random variable with mean $T(t)x(0)$ and covariance $C(t)$.

Example 1. $\mathcal{K} = L_2[0, 1]$, $A = \frac{d^2}{dx^2}$ with Dirichlet boundary conditions. B is identity.

$$du(t, x) = \frac{1}{2}u_{xx}dt + d\beta(t, x)$$

The question then is the integrability of

$$\int_0^t \int_0^1 \int_0^1 |p(s, x, y)|^2 dxdyds = \int_0^t \int_0^1 p(2s, x, x)dxds \simeq \int_0^t \frac{ds}{\sqrt{s}} < \infty$$

Example 2. If we replace one dimensional $A = \frac{d^2}{dx^2}$ by the two dimesional $\frac{1}{2}\Delta$ with Dirichlet boundary conditions in a boundend domain D then $\int_0^t \int_D p(2s, x, x)dxds = \infty$. This means we need to smoothen the noise a little bit with a compact B.

Assuming we can define every thing, we end up with a transition probability $q(t, x, dy)$ given by a Gaussian measure with mean $T(t)x$ and covariance $C(t)$. This will satisfy the Chapman-Kolmogorov equations. The process is clearly Feller, i.e the map $x \to q(t, x, dy)$ is weakly continuous for each fixed $t > 0$.

We now look at the question of when the process $x(t)$ is almost surely continuous as a function of t. The argument we use in finite dimensions, i.e. decomposing

$$\int_0^t T(t-s)Bd\beta(s) = T(t)\int_0^t T(-s)Bd\beta(s)$$

and using the martingale property will not work, since $T(t)$ can be nasty for $t < 0$. We use a trick of representing

$$w(t) = \int_0^t T(t-s)Bd\beta(s)$$

as

$$w(t) = \int_0^t T(t-\tau)(t-\tau)^{-(1-\alpha)}u(\tau)d\tau$$

where

$$u(\tau) = c(\alpha)\int_0^\tau T(\tau-s)(\tau-s)^{-\alpha}Bd\beta(s)$$

and $c(\alpha) = [\Gamma(\alpha)\Gamma(1-\alpha)]^{-1}$. We can check it

$$\begin{aligned}
c(\alpha)&\int_0^t T(t-\tau)(t-\tau)^{-(1-\alpha)}u(\tau)d\tau \\
&= c(\alpha)\int_0^t\int_0^\tau T(t-s)(t-\tau)^{-(1-\alpha)}(\tau-s)^{-\alpha}d\tau Bd\beta(s) \\
&= \int_0^t T(t-s)[c(\alpha)\int_s^t (t-\tau)^{-(1-\alpha)}(\tau-s)^{-\alpha}d\tau]Bd\beta(s) \\
&= \int_0^t T(t-s)Bd\beta(s)
\end{aligned}$$

If we assume that

$$\int_0^t s^{-2\alpha}T(s)BB^*T(s)^*ds < \infty$$

then we can control $E[\|u(\tau)\|^2]$. Just as in the scalar case one can show that

$$E[\|z\|^p] \le C_p E[\|z\|^2]^{\frac{p}{2}}$$

We assume that p is an integer. This amounts to observing that in some basis where X_i are independent

$$E[[\sum_i X_i^2]^p] \le E[[\sum_{i_1,i_2,\ldots,i_p} X_{i_1}^2 X_{i_2}^2 \ldots X_{i_p}^2]] \le C_p[\sum_{i_1,i_2,\ldots,i_p} E[X_{i_1}^2]E[X_{i_2}^2]\ldots E[X_{i_p}^2]]$$

and we can control $E[\|u(\tau)\|^p]$. We use the representation

$$w(t) = \int_0^t T(t-s)(t-s)^{-(1-\alpha)}u(s)ds$$

to prove the continuity of $w(t)$. Take $\epsilon > 0$ and truncate

$$w_\epsilon(t) = \int_0^{t-\epsilon} T(t-s)(t-s)^{-(1-\alpha)}u(s)ds$$

$w_\epsilon(t)$ is well defined and continuous for $t \geq \epsilon$ and

$$\|w_\epsilon(t) - w(t)\| \leq C\int_{t-\epsilon}^t (t-s)^{-(1-\alpha)}\|u(s)\|ds$$

is easily controlled if p is taken large enough. Then by Fubini's theorem $\|u(s)\|$ would be in L_p almost surely and $\|w_\epsilon(t) - w(t)\|$ would be uniformly small.

Strong Feller Property. The strong Feller property states that the map $x \to q(t,x,A)$ is continuous for every Borel set A and every $t > 0$. This impies that the semigroup T_t maps the space of bounded measurable functions $B(\mathcal{H})$ into the space of bounded continuous $C(\mathcal{H})$. Assuming that $\mathcal{H}$ is separble this implies that the measures $q(t,x,dy)$ are all dominated by a single measure μ with respect to which there is a density $\bar{q}(t,x,y)$. In particular any invariant measure, if it exists is absolutely continuous with respect to μ and this makes the ergodic theory manageable. Because the Gaussian measures with different means a_1, a_2 and common covariance C in a Hilbert space are mutually absolutely continuous if and only if $a_1 - a_2 = C^{\frac{1}{2}}x$ for some $x \in \mathcal{H}$, the condition for strong Feller property reduces to the condition that for positive t the range of T_t is contained in the range of $[C(t)]^{\frac{1}{2}}$. i.e. given $t > 0$ and $x \in \mathcal{H}$, there is $y \in \mathcal{H}$ such that

$$T_t x = [C(t)]^{\frac{1}{2}}y$$

Null Controllability. A linear system

$$dx(s) = Ax(s)ds + Bu(s)ds$$

is null controllable in $[0,t]$ if one can find for any given $x \in \mathcal{H}$ a function $u(s) \in L_2[[0,t];\mathcal{K}]$ such that the solution of

$$dx(s) = Ax(s)ds + Bu(s)ds$$

with $x(0) = x$ ends up at $x(t) = 0$. Can be driven to zero with a square integrable control in time t. If that is possible we can try to minimize $\int_0^t \|u(s)\|^2 ds$ over all such controls. A standard perturbation argument tells us that the minimizer $u(\cdot)$ exits, satisfies

$$\int_0^t < u(s), w(s) > ds = 0$$

for every w with

$$\int_0^t T(t-s)Bw(s)ds = 0$$

In particular

$$u(s) = B^*T^*(t-s)y$$

for some $y \in \mathcal{H}$ and

$$T(t)x + \int_0^t T(t-s)BB^*T^*(t-s)yds = 0$$

or

$$T(t)x = C(t)y$$

But

$$\int_0^t \|u(s)\|^2 ds =< y, C(t)y >= \|[C(t)]^{\frac{1}{2}}y\|^2$$

Proves that strong Feller is equivalent to Null Controllability.

Note that if B has a bounded inverse then the choice of $u(s) = -\frac{1}{t}B^{-1}T(s)x$ makes

$$x(t) = T(t)x + \int T(t-s)Bu(s) = T(t)x - \frac{1}{t}\int_0^t T(t)xds = 0$$

Shows that if B has a bounded inverse then it is controllable.

Finally if A is dissipative, i.e. $\|T(t)x\| \to 0$ for every $x \in \mathcal{H}$, then $q(t,x,dy)$ has a limit $Q(dy)$, as $t \to \infty$, which is Gaussian with mean 0 and covariance

$$C = \int_0^\infty T(s)BB^*T^*(s)ds$$

provided $Tr\ C = \int_0^\infty Tr\ [T(s)BB^*T^*(s)]ds < \infty$. Since

$$C = T(t)^*CT(t) + \int_0^t T(s)BB^*T^*(s)ds$$

Q is invariant.