**1. Gaussian Model.**

We will be dealing with Gaussian Models. The central object is the Cameron-Martin space which will be a real Hilbert Space $\mathcal{H}$. There is a probability space $(X, \mathcal{F}, P)$, that can be thought of as a vector space containing $\mathcal{H}$ and there is a linear map $h \to \xi_h$ of $\mathcal{H}$ in to $L_2(X)$ that represents $\mathcal{H}$ isometrically as Gaussian random variables with mean 0. In addition $\mathcal{H}$ acts by translations on $\tau_h : x \to x + h$ on $X$. These actions are quasi invariant and if we denote $P\,\tau_h^{-1}$ by $P_h$ then $P_h << P$ and

$$\frac{dP_h}{dP} = \exp[\xi_h(x) - \frac{1}{2} < h, h >]$$

One can assume the completeness of the model which amounts to the assumption that $\mathcal{F}$ is generated by $\{\xi_h : h \in \mathcal{H}\}$.

As a probability space we have the $L_p$ spaces on $(X, \mathcal{F}, P)$.

There is a special basis for $L_2$. We can choose a complete orthonormal basis for $\mathcal{H}$ and the corresponding $\xi_j$ give a sequence of i.i.d. random variables that are standard Gaussians, and our probability space is essentially equivalent to the product measure of these standard Gaussians. In individual coordinates we have the basis of Hermite polynomials $H_i(\xi_j)$. For any function $p(\cdot) : \{1, 2, \cdots\} \to \{0, 1, 2 \cdots\}$ that has only finitely many nonzero entries we can asociate the functions

$$\overline{H}_{p(\cdot)} = \Pi_i H_{p(i)}(\xi_i)$$

and they form a basis for the tensor product. This basis comes naturally graded by 'degree' $n = \sum_i p(i)$. The subspaces of $\mathcal{K}_n$ degree $n$ are natural closed subspaces that are mutually orthogonal. Degree 0 space is just the one dimensionl space of constants, degree 1 is a space is canonically isomorphic to $\mathcal{H}$ and the orthogonal sum of all subspaces of degree $n$ or less is just the space of polynomials in the Gaussian variables of degree less than or equal to $n$.

**Analysis on the Gaussian Space.** The notion of gradients have to be carefully defined. Thre is no concept of continuity because we have imposed no topology on $X$. Even if we could, the natural functions that we want to consider, like solutions to Ito's equations, will not be continuous. So we have to live with smoothness in the sense of Sobolev spaces. We are in infinite dimensions and there is no imbedding theorem and consequently we can have infinite smoothness in the Sobolev sense with out gaining any regularity in the sense of continuity.

**Gradients.** We can define the notion of $D_h$, the directional derivative in the direction of $h \in \mathcal{H}$ by demanding that

$$(D_h f)(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon h) - f(x)}{\varepsilon}$$

exist in $L_p$. Then look for a representation

$$(D_h f)(x) =< (Df)(x), h >$$

for some function on $X$ with values in $\mathcal{H}$. Then $Df : X \to \mathcal{H}$ is called the gradient.

$$\|Df\|_p^p = \int_X ||(Df)(x)||_{\mathcal{H}}^p \, P(dx)$$

$$\|f\|_{1,p} = \|Df\|_{L_p} + \|f\|_{L_p}$$

**Integration by Parts.** We can start with the identity

$$\int_X f(x + \varepsilon h) P(dx) = \int_X f(x) \exp[\varepsilon \xi_h(x) - \frac{\varepsilon^2}{2} < h, h >] P(dx)$$

and upon differentiation with respect to $\varepsilon$ at $\varepsilon = 0$ we get

$$\int_X (D_h f)(x) P(dx) = \int_X \xi_h(x) f(x) \, P(dx)$$

**Ornstein-Uhlenbeck Semigroup.** The following family of operators defines a semigroup of contractions on any $L_p(X)$.

$$(P_t f)(x) = \int_X f(e^{-t}\, x + \sqrt{1 - e^{-2t}}\; y) P(dy)$$

To see the contraction part we remark that if $x$ and $y$ are independent mean 0 Gaussians with the same distribution then $ax + by$ is again a Gaussian with the same distribution provided $a^2 + b^2 = 1$. The rest is just Holder's inequality. The semigroup corresponds to a Markov Process with values in $X$, but the trajectories of this proces will play no role at the momemt.

**Generator of the Ornstein-Uhlenbeck Semigroup.** From the structure of the independence of the Gaussian random variables corresponding to different mutually orthogonal elements in $\mathcal{H}$, it is clear that

$$\overline{H}_{p(\cdot)} = \Pi_i H_{p(i)}(\xi_i)$$

is an eigenfunction for the OU semigroup

$$P_t \overline{H}_{p(\cdot)} = e^{-n\,t} \overline{H}_{p(\cdot)}$$

where $n$ is the 'degree'. The Generator $\mathcal{L}$ of the OU process is defined by

$$\mathcal{L}\overline{H}_{p(\cdot)} = -n\,\overline{H}_{p(\cdot)}$$

Let us remark that in $L_2$ the semigroup is self adjoint, and the complete spectral resolution is provided above. We have the resolvent operator

$$(I - \mathcal{L})^{-1} = \int_0^\infty e^{-t} P_t dt$$

as well as

$$\Lambda = (1 - \mathcal{L})^{-\frac{1}{2}} = \int_0^\infty e^{-t} \frac{1}{\sqrt{\pi\,t}} P_t dt$$

that are again contarctions in every $L_p$ for $p \geq 1$.

Multiplier Lemma. Consider the operator $\mathcal{A}$ that acts on $\mathcal{K}_n$ by multiplication by the scalar $(\frac{a+n}{b+n})^{\pm\frac{1}{2}}$. Then for $a, b > 0$, $\mathcal{A}$ is bounded on every $L_p$.

**Proof.** We can assume that $b > a$. We write

$$\left( \frac{a+n}{b+n} \right)^{\pm\frac{1}{2}} = \left( 1 - \frac{b-a}{b+n} \right)^{\pm\frac{1}{2}} = \sum c_j (b-a)^j (b+n)^{-j}$$

By using the obvious estimates on $c_j$ and the fact that $b(b-\mathcal{L})^{-1}$ is a contraction we are done.

**Theorem (Calderon-Zygmund).** For every $1 < p < \infty$ there is a constant $c_p$, such that

$$\|\Lambda f\|_{1,p} \leq c_p \|f\|_{L_p}.$$

**Proof.** Since $\Lambda$ is a contraction in every $L_p$ we only need to estimate $\|D\Lambda f\|_{L_p}$. To this end we calculate 'explicitly' $D_h\Lambda f$. There is a singularity at $t=0$ that we ignore for the moment.

$$\begin{aligned}
D_h\Lambda f &= D_h\int_0^\infty \frac{e^{-t}}{\sqrt{\pi\, t}} P_t f dt\\
&= D_h\int_0^\infty \frac{e^{-t}}{\sqrt{\pi\, t}}\int_X f(e^{-t}x+\sqrt{1-e^{-2t}}\ y)P(dy)dt\\
&= \int_0^\infty \frac{e^{-t}}{\sqrt{\pi\, t}}\int_X e^{-t}(D_h f)(e^{-t}x+\sqrt{1-e^{-2t}}\ y)P(dy)dt\\
&= \int_0^\infty \int_X \frac{e^{-t}}{\sqrt{\pi\, t}}\frac{e^{-t}}{\sqrt{1-e^{-2t}}}(D_h^y f)(e^{-t}x+\sqrt{1-e^{-2t}}\ y)P(dy)dt\\
&= \int_0^\infty \int_X \frac{e^{-t}}{\sqrt{\pi\, t}}\frac{e^{-t}}{\sqrt{1-e^{-2t}}}\xi_h(y) f(e^{-t}x+\sqrt{1-e^{-2t}}\ y)P(dy)dt
\end{aligned}$$

We double up the integral from $[-\infty,\infty]$ and use the invariance of $P$ under the map $y\to -y$ as well as the oddness of $\xi_h(y)$. Then with $\varepsilon(t)=\frac{t}{|t|}$,

$$\begin{aligned}
(D_h\Lambda f)(x) &= \frac{1}{2}\int_{-\infty}^\infty \int_X \frac{e^{-|t|}\varepsilon(t)}{\sqrt{\pi\, |t|}}\frac{e^{-|t|}}{\sqrt{1-e^{-2|t|}}}\xi_h(y) f(e^{-|t|}x+\varepsilon(t)\sqrt{1-e^{-2|t|}}\ y)P(dy)dt\\
&= \langle \xi_h(\cdot)\,, u(x\,,\cdot)\rangle_{L_2(P)}
\end{aligned}$$

In other words with the identification of $\mathcal{K}_1$ with $\mathcal{H}$,

$$(D\Lambda f)(x) = P_1 u(x\,,\cdot)$$

We need the following lemma.

**Lemma.** If we denote by $P_1$ the orthogonal projection in $L_2(P)$ on to the subspace $\mathcal{K}_1$ of dgeree 1, then for any $1<p<\infty$, there is a finite constant $C_p$ such that

$$\|P_1 f\|_{L_p(P)} \le C_p \|f\|_{L_p(P)}$$

**Proof:** For $p\ge 2$ since $P_1 f$ is Gaussian

$$\|P_1 f\|_{L_p(P)} \le C_p\,\|P_1 f\|_{L_2(P)} \le C_p\|f\|_{L_2(P)} \le C_p\,\|f\|_{L_p(P)}$$

The inequality for $1<p\le 2$ is derived by duality, with $C_p=C_q$. Actually a similar estimate is valid for the projection $P_n$ on to $\mathcal{K}_n$, the subspace of 'degree' $n$. The proof depends on the following elementary computation.

**Lemma.** There is a constant $C(n,p)$ such that for $f \in \cup_{j=1}^{n} \mathcal{K}_j$

$$\|f\|_{L_p(P)} \leq C(n,p)\|f\|_{L_2(P)}$$

**Proof:** It is sufficient to show the existence of a constant $C_n$ such that for $f \in \cup_{j=1}^{n} \mathcal{K}_j$

$$\|f^2\|_{L_2(P)} \leq C_n \, \|f\|_{L_2(P)}^2$$

To this end we can write

$$f = \sum_{\deg \ p(\cdot) \leq n} A_{p(\cdot)}[\otimes_j \overline{H}^j_{p(j)}]$$

where $\overline{H}^j_r$ is the normalized Hermite polynomial of degree $r$ in the variable $x_j$. Then

$$\begin{aligned} f^2 &= \sum_{\deg \ p(\cdot) \leq n} \ \sum_{\deg \ q(\cdot) \leq n} A_{p(\cdot)} \, A_{q(\cdot)} \otimes_j \overline{H}^j_{p(j)} \overline{H}^j_{q(j)} \\ &= \sum_{\deg \ p(\cdot) \leq n} \ \sum_{\deg \ q(\cdot) \leq n} A_{p(\cdot)} \, A_{q(\cdot)} \otimes_j \big[ \sum_r a(p(j), q(j), r) \overline{H}^j_r \big] \\ &= \sum_{\deg \ r(\cdot) \leq 2n} B_{r(\cdot)} \otimes_j \overline{H}^j_{r(j)} \end{aligned}$$

with

$$\begin{aligned} B_{r(\cdot)} &= \sum_{p(\cdot), q(\cdot)} A_{p(\cdot)} \, A_{q(\cdot)} \prod_j a(p(j), q(j), r(j)) \\ &= \sum_{p(\cdot), q(\cdot)} A_{p(\cdot)} \, A_{q(\cdot)} \, C(p(\cdot) \, , q(\cdot) \, , r(\cdot)) \end{aligned}$$

Note that $a(p,q,r) = 0$ unless $p + q \geq r$. Therefore for each $p(\cdot)$ and $q(\cdot)$ of degree $n$ there are atmost some $k(n)$ terms that are nonvanishing in the last sum and there is a uniform bound $C_n$ on on $|C(\cdot \, , \cdot \, , \cdot)|$. Therefore

$$\|f^2\|_{L_2(P)}^2 = \sum_{r(\cdot)} |B_{r(\cdot)}|^2 \leq C_n^2 \, k(n) \big( \sum_{p(\cdot)} |A_{p(\cdot)}|^2 \big)^2 = C_n^2 \, k(n) \|f\|_{L_2(P)}^4.$$

Proceeding with our proof, because

$$\|(D\Lambda f)(x)\|_{\mathcal{H}} = \|P_1 u(x \, , \cdot)\|_{L_2(P)},$$

from the previous lemma,

$$\int_X \|(D\Lambda f)(x)\|_{\mathcal{H}}^p P(dx) \leq C_p \int_X \int_X |u(x \, , y)|^p P(dx) P(dy)$$

After the substitution $|t| = \log\sec\theta$, $u(x\,,y)$ can be represented as

$$u(x\,,y) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{-|t|}\varepsilon(t)}{\sqrt{\pi\,|t|}} \frac{e^{-|t|}}{\sqrt{1-e^{-2|t|}}} f(e^{-|t|}x + \varepsilon(t)\sqrt{1-e^{-2|t|}}\;y)dt$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\sin\theta\, x + \cos\theta\, y)k(\theta)\, d\theta$$

with

$$k(\theta) = \frac{1}{2\sqrt{\pi}} \frac{\cos\theta\,\varepsilon(\theta)}{\sqrt{\log\sec\theta}} = c\,k_0(\theta) + k_1(\theta)$$

Here, $c$ is a positive constant, $k_0(\theta)$ is the singular kernel representing the Hilbert transform and $k_1(\cdot)$ is a bounded function. If we write the corresponding decomposition

$$u(x\,,y) = u_0(x\,,y) + u_1(x\,,y)$$

for $p \geq 1$, since we have a $L_\infty$ bound on $k_1(\cdot)$,

$$\int_X\int_X \big|u_1(x\,,y)\big|^p P(dx)P(dy)$$
$$= \int_X\int_X \left|\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\cos\theta\, x + \sin\;\theta\, y)k_1(\theta)\, dt\right|^p P(dx)P(dy)$$
$$\leq C\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_X\int_X \left|f(\cos\theta\, x + \sin\;\theta\, y)\right|^p P(dx)P(dy)dt$$
$$= C\int_X \left|f(\cos\theta\, x + \sin\;\theta\, y)\right|^p P(dx)$$

by the rotational invariance of the product Gaussian. As for the Hilbert transform term $u_0(x\,,y)$ we can write

$$u_0(\cos\theta\, x + \sin\theta\, y, -\sin\theta\, x + \cos\theta\, y) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\cos\theta'\, x + \sin\;\theta'\, y)k_o(\theta - \theta')d\theta'$$

and

$$\int_X\int_X \big|u_0(x\,,y)\big|^p P(dx)P(dy)$$
$$= \int_X\int_X \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left|u_0(\cos\theta\, x + \sin\theta\, y, -\sin\theta\, x + \cos\theta\, y)\right|^p P(dx)P(dy)d\theta$$
$$\leq C_p\int_X\int_X \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left|f(\cos\theta\, x + \sin\theta\, y)\right|^p P(dx)P(dy)d\theta$$
$$= C_p\int_X \big|f(x)\big|^p P(dx)$$

and we are done.

**Remark 1.** We have ignored the singularity. From a technical viewpoint we should have cut off near the singularity and passed to the limit with uniform bounds.

**Remark 2.** Since $\|\Lambda f\|_{L_p(P)} \leq \|f\|_{L_p(P)}$ we have $\|\Lambda f\|_{1,p} \leq C_p \|f\|_{L_p(P)}$.

**Remark 3.** It is not clear that if $\|f\|_{1,p} < \infty$,

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[f(x + \varepsilon\, h) - f(x)]$$

necessarily make sense in $L_p(P)$. Why is $f(x + \varepsilon\, h)$ in $L_p(P)$? It is therefore better to work with $\cap_p L_p(P)$ and complete in the various $\| \cdot \|_{r,p}$ norms.

**Remark 4.** We can consider functions $f$ with values in a Hilbert Space $\mathcal{V}$. If we take a model with $\mathcal{V}$ for its Cameron-Martin space, then $f$ can be identified with scalar functions $f(x, v)$ that are linear in $v$. The new $D$ has two components, in the original $\mathcal{H}$ directions and the new $\mathcal{V}$ directions. The new $\Lambda$ views the functions differently, essentially increasing the degree by 1, because of tensoring with the linear functions, that have one extra degree. Because of the multiplier theorem the two are essentially equivalent on $L_p$ spaces. The result for $\mathcal{V}$ valued functions can be read off from the result on scalar functions of both sets of variables, although our interest is only in functions that are linear in the second set.