Lemma 1. Consider a positive definite symmetric matrix A.

$$(Det\ A)^{-\frac{1}{2}} = c_n \int_{S^{n-1}} \frac{ds}{[< s, As >]^{\frac{n}{2}}}$$

Proof:

$$\begin{aligned}(Det\ A)^{-\frac{1}{2}} &= (2\pi)^{\frac{n}{2}} \int_{R^n} e^{-\frac{<x,Ax>}{2}} dx \\ &= (2\pi)^{\frac{n}{2}} \int_{S^{n-1}} e^{-r^2 \frac{<s,As>}{2}} r^{n-1} ds dr \\ &= c_n \int_{S^{n-1}} \frac{ds}{[< s, As >]^{\frac{n}{2}}}\end{aligned}$$

It is therefore sufficient to estimate for each k,

$$\sup_{x:\|x|\|=1} E[< x, Ax >^{-k}] \le C_k$$

to yield an estimate of the form

$$E[(Det\ A)^{-k}] \le B_k$$

Weak formulation. With out estimates.

Let us note that the Malliavin covariance $A(t)$ has the form

$$A(t) = \int_0^t B(s,t,\omega) a(x(s)) B^*(s,t,\omega) ds$$

with $B(s,t,\omega)$ being the Jacobian of the map $R^n \to R^n$ that maps the initial point $x = x(s,\omega)$ in R^n to $x(t) \in R^n$. The problem with $A(t)$ is that $B(s,t,\omega)$ is NOT progressively measurable. Moreover intrinsically $A(t)$ is a quadratic form on the cotangent space at $x(t,\omega)$. We can instead look at $B(0,t,\omega)^{-1} A(t) B(0,t,\omega)^{-1\ *}$. Since $Det\ B(0,t,\omega)$ and its inverse will have moments of all orders, we can try to show that

$$C(t) = B(0,t,\omega)^{-1} A(t) B(0,t,\omega)^{-1\ *}$$

has a nice inverse. $C(t)$ has a better representation, $B(0,s,\omega)$ being progressively measurable.

$$C(t) = \int_0^t B(0,s,\omega)^{-1} a(x(s)) B(0,s,\omega)^{-1\ *} ds$$

Suppose $C(1)x = 0$ for some $x = \{x_i\}$. Then denoting by $h_{i,j}(s,\omega) = (B(0,s,\omega)^{-1})_{i,j}$ we have

$$\sum_{i,j} h_{i,j}(s,\omega) x_i \sigma_{j,k}(x(s)) \equiv 0$$

for $k = 1, 2, \ldots n$. If we think of $\sigma_{j,k}(x)$ as vector fields $X_k(x)$, then

$$\sum_i x_i q_{i,k}(s, \omega) \equiv 0$$

for $k = 1, 2, \ldots n$, where

$$\sum_i x_i q_{i,k}(s, \omega) =< x, h(s, \omega) X_k(x(s)) >$$

Now

$$\xi(s) =< x, h(s, \omega) X_k(x(s)) >$$

is a semi-martingale and we can compute the Stratonovich differential

$$\begin{aligned} d \circ (h(s, \omega) X_k(x(s)) &= [d \circ h(s, \omega)] X_k(x(s)) + h(s, \omega)[d \circ X_k(x(s))] \\ &= \sum_r e_{k,r}(s, \omega) \circ d\beta_r(s) + e_{k,0}(s, \omega) ds \end{aligned}$$

One can compute easily

$$e_{k,r}(s) = h(s)[-X_r X_k + X_k X_r](x(s)) = h(s)[X_k, X_r](x(s))$$

and

$$e_{k,0}(s, \omega) = h(s)[-X_0 X_k + X_k X_0](x(s)) = h(s)[X_k, X_0](x(s))$$

One knows from the uniqueness in Doob-Meyer decomposition that if

$$d\xi = dM(t) + b(t)dt \equiv 0$$

then $M(t) \equiv 0$ and $b(t) \equiv 0$. Moreover if

$$dM(t) = \sum e_j(s) d\beta_j(s)$$

Therefore $e_{k,r}$ are equal to 0. The induction proceeds. By Blumenthal zero-one law if the determinant is zero for a positive time, then it is so with probability one and there is a deterministic direction in which it is degenerate. That direction is orthogonal to all the vectors generated by all the Lie brackets.

Quantitative version.

We will estimate $E[X^{-k}]$ by estimating $E[e^{-\lambda X}]$ and integrating

$$E[X^{-k}] = \frac{1}{\Gamma(k)} \int E[e^{-\lambda X}] \lambda^{k-1} \, d\lambda$$

We will fix M a bound on σ and b as well as the time interval $[0, T]$. $C(T, M)$ will stand for a constant that may depend on M and T but independent of λ.

Lemma 1. Let $\xi(t)$ be a stochastic integral

$$\xi(t) = x + \int_0^t \sigma(s)\cdot d\beta(s) + \int_0^t b(s)ds$$

such that

$$|\sigma(s)| \leq M \quad \text{and} \quad |b(s)| \leq M$$

Then for any $\lambda \geq 0$,

$$E\left[\exp\big[-\frac{\lambda^2}{4}\int_0^t \xi^2(s)ds + \frac{\lambda}{4M}\int_0^t \sigma^2(s)ds\big]\right] \leq C(t)$$

Proof. Consider the function

$$U(t\,,x) = \exp\big[-\frac{\lambda x^2}{4M}\tanh\lambda Mt + \frac{1}{4}F(\lambda Mt) + \frac{1}{2}t\big]$$

where

$$F(x) = \int_0^x [1-\tanh x]dx = x - \log\cosh x \leq \log 2$$

Then

$$\begin{aligned}
U_x &= U[-\frac{\lambda x}{2M}\tanh\lambda Mt\,]\\
U_{xx} &= U[\,\frac{\lambda^2x^2}{4M^2}\tanh^2\lambda Mt - \frac{\lambda}{2M}\tanh\lambda Mt\,]\\
U_t &= U[-\frac{\lambda^2x^2}{4}\text{sech}^2\lambda Mt + \frac{\lambda M}{4}(1-\tanh\lambda Mt) + \frac{1}{2}\,]
\end{aligned}$$

$$\begin{aligned}
&\sup_{\substack{0\leq\sigma\leq M\\ |b|\leq M}}\big[\frac{\sigma^2}{2}U_{xx} + b\,U_x - \frac{\lambda^2x^2}{4}U + \frac{\lambda\,\sigma^2}{4M}U\big]\\
&\leq U\big[\frac{\lambda^2x^2}{8}\tanh^2\lambda Mt - \frac{\lambda^2x^2}{4} + \frac{\lambda\,M}{4}(1-\tanh\lambda Mt) + \frac{\lambda|x|}{2}\tanh\lambda Mt\,\big]\\
&\leq U\big[\frac{\lambda^2x^2}{4}\tanh^2\lambda Mt - \frac{\lambda^2x^2}{4} + \frac{\lambda\,M}{4}(1-\tanh\lambda Mt) + \frac{1}{2}\,\big]\\
&= U_t
\end{aligned}$$

Therefore

$$Z_t = U(T-t\,,\xi(t))\exp[-\frac{\lambda^2}{4}\int_0^t \xi^2(s)ds + \frac{\lambda}{4M}\int_0^t \sigma^2(s)ds]$$

is a super martingale.

$$E[Z_T] \leq E[Z_0]$$

Let $\xi(t)$ as before be

$$\xi(t) = x + \int_0^t < e(s).d\beta(s) > + \int_0^t b(s)ds$$

and denote by

$$\eta(t) = x + \int_0^t < e(s).d\beta(s) > ds$$

$$B(t) = \int_0^t b(s)ds$$

so that

$$\xi(t) = x + \eta(t) + B(t)$$

Assume that

$$\sigma(s) = \|e(s)\| \leq M \qquad \text{a.e.}$$

and

$$|b(s)| \leq M \qquad \text{a.e.}$$

Lemma 2. We have

$$E\left[\exp\big[\lambda \int_0^T |\eta(s)|ds - \lambda^2 T^2 \int_0^T \sigma^2(s)ds\big]\right] \leq C.$$

Proof. Apply Doob's inequality to the non negative martingale

$$\exp\big[\lambda\, \eta(t) - \frac{\lambda^2}{2}\int_0^t \sigma^2(s)ds\big]$$

to get

$$P\left[\exp\big[\sup_{0\leq t\leq T}[\lambda\, \eta(t) - \frac{\lambda^2}{2}\int_0^t \sigma^2(s)ds]\big] \geq \ell\right] \leq \frac{1}{\ell}$$

and for $\lambda \geq 0$, replacing λ by $2\,\lambda$,

$$P\left[\exp\big[2\lambda \sup_{0\leq t\leq T} \eta(t) - 2\,\lambda^2 \int_0^T \sigma^2(s)ds\big] \geq \ell\right] \leq \frac{1}{\ell}$$

This leads to

$$E\left[\exp\big[\lambda \sup_{0\leq t\leq T} |\eta(t)| - \lambda^2 \int_0^T \sigma^2(s)ds\big]\right] \leq C$$

and

$$E\left[\exp\big[\frac{\lambda}{T}\int_0^T |\eta(s)|ds - \lambda^2 \int_0^T \sigma^2(s)ds\big]\right] \leq C$$

and replacing λ by $\lambda\, T$,

$$E\left[\exp\big[\lambda \int_0^T |\eta(s)|ds - \lambda^2\, T^2 \int_0^T \sigma^2(s)ds\big]\right] \leq C$$

Lemma 3. For any $\lambda \geq 0$,

$$E\Big[\exp\big[-4M\lambda^2T^2\int_0^T \xi^2(s)ds - \frac{\lambda}{2}\int_0^T |\xi(s)|ds + \frac{\lambda^2T^2}{2}\int_0^T \sigma^2(s)ds + \frac{\lambda}{2}\int_0^T |B(s)|ds\big]\Big]$$
$$\leq C(T)$$

Proof: We have by the earlier lemma, for any $\mu > 0$

$$E\left[\exp\big[-\frac{\mu^2}{4}\int_0^T \xi^2(s)ds + \frac{\mu}{4M}\int_0^T \sigma^2(s)ds\big]\right] \leq C(T)$$

By Schwarz's inequality, in combination with Lemma 2, with the choice of $\mu = 4M\lambda^2T^2$

$$E\left[\exp\big[-4M\lambda^2T^2\int_0^T \xi^2(s)ds + \frac{\lambda^2T^2}{2}\int_0^T \sigma^2(s)ds + \frac{\lambda}{2}\int_0^T |\eta(s)|ds\big]\right] \leq C(T)$$

This in turn implies

$$E\Big[\exp\big[-4M\lambda^2T^2\int_0^T \xi^2(s)ds - \frac{\lambda}{2}\int_0^T |\xi(s)|ds + \frac{\lambda^2T^2}{2}\int_0^T \sigma^2(s)ds + \frac{\lambda}{2}\int_0^T |B(s)|ds\big]\Big]$$
$$\leq C(T)$$

If X and Y are two random variables such that

$$E\left[\exp[-aX + bY]\right] \leq C$$

then

$$E\left[\exp[-\frac{a}{2}X]\right] = E\left[\exp[-\frac{a}{2}X + \frac{b}{2}Y - \frac{b}{2}Y]\right] \leq \sqrt{C}\left[E\left[\exp[-bY]\right]\right]^{\frac{1}{2}}$$

Therefore

$$E\Big[\exp\big[-2M\lambda^2T^2\int_0^T \xi^2(s)ds - \frac{\lambda}{4}\int_0^T |\xi(s)|ds\big]\Big]$$
$$\leq C(T)E\Big[\exp\big[-\int_0^T \frac{\lambda^2T^2}{2}\sigma^2(s)ds - \frac{\lambda}{2}\int_0^T |B(s)|ds\big]\Big]^{\frac{1}{2}}$$

$$2M\lambda^2T^2\int_0^T \xi^2(s)ds + \frac{\lambda}{4}\int_0^T |\xi(s)|ds$$
$$\leq 2M\lambda^2T^2\int_0^T \xi^2(s)ds + \frac{\lambda\sqrt{T}}{4}\left[\int_0^T \xi^2(s)ds\right]^{\frac{1}{2}}$$
$$\leq 2M\lambda^2T^2\int_0^T \xi^2(s)ds + 2\lambda^2T\int_0^T \xi^2(s)ds + \frac{1}{32}$$
$$\leq 2\lambda^2T(1+MT)\int_0^T \xi^2(s)ds + \frac{1}{32}$$

Therefore

$$E\Big[\exp\big[-2\lambda^2 T(1+MT)\int_0^T \xi^2(s)ds\big]\Big]$$
$$\leq C(T)E\Big[\exp\big[-\frac{\lambda^2T^2}{2}\int_0^T\sigma^2(s)ds-\frac{\lambda}{2}\int_0^T|B(s)|ds\big]\Big]^{\frac{1}{2}}$$
$$\leq C(T)E\Big[\exp\big[-\frac{\lambda^2T^2}{2}\int_0^T\sigma^2(s)ds\big]\Big]^{\frac{1}{2}}$$

and

$$E\Big[\exp\big[-2\lambda^2 T(1+MT)\int_0^T \xi^2(s)ds\big]\Big]$$
$$\leq C(T)E\Big[\exp\big[-\frac{\lambda}{2}\int_0^T|B(s)|ds\big]\Big]^{\frac{1}{2}}$$

$$\int_0^\infty e^{-Ax^2}x^k dx=\frac{1}{2}\int_0^\infty e^{-Ay}y^{\frac{k-1}{2}}dy=\frac{\Gamma(\frac{k+1}{2})}{2}A^{-\frac{k+1}{2}}$$

$$\int_0^\infty\sqrt{(u(x))}\,x^k dx\leq\int_0^1\sqrt{(u(x))}\,x^k dx+\int_1^\infty\sqrt{(u(x))}\,x^k dx$$
$$\leq\Big[\int_0^1 u(x)\,x^{2k}dx\Big]^{\frac{1}{2}}+\Big[\int_1^\infty\frac{dx}{x^2}\Big]^{\frac{1}{2}}\Big[\int_1^\infty u(x)x^{2k+2}dx\Big]^{\frac{1}{2}}$$
$$=\Big[\int_0^\infty u(x)\,x^{2k}dx\Big]^{\frac{1}{2}}+\Big[\int_0^\infty u(x)\,x^{2k+2}dx\Big]^{\frac{1}{2}}$$

Therefore

$$E\Big[\big[\int_0^T\xi^2(s)ds\big]^{-\frac{k+1}{2}}\Big]\leq C(T,k)E\left[1+\left[\int_0^T\sigma^2(s)ds\right]^{-\frac{2k+3}{2}}\right]^{\frac{1}{2}}$$

and

$$E\Big[\big[\int_0^T\xi^2(s)ds\big]^{-\frac{k+1}{2}}\Big]\leq C(T,k)E\left[1+\left[\int_0^T|B(s)|ds\right]^{-(2k+3)}\right]^{\frac{1}{2}}$$

The final step is to estimate

$$E\left[\left[\int_0^T|B(s)|ds\right]^{-2k}\right]$$

in terms of

$$E\left[\left[\int_0^T|b(s)|^2ds\right]^{-k}\right]$$

where

$$B(t) = x + \int_0^t b(s)ds$$

Should depend on the simple estimate (Sobolev)

$$\|b\|_{L_2[0,T]} \leq C_T(\|B\|_{L_1[0,T]})^a(\|b\|_{H^p_\alpha[0,T]})^{1-a}$$

First if we consider

$$\Theta^2 = \int_0^T \int_0^T \frac{|b(t) - b(s)|^2}{|t-s|^{\frac{7}{4}}} dtds$$

then

$$\Theta = \|b\|_{H^p_\alpha[0,T]}$$

with $p = 2$ and $\alpha = \frac{3}{8}$.

$$E[\Theta^k] \leq C(T, k)$$

and one can get

$$\|b\|_{L_2[0,T]} \leq C_T\Theta^{1-a}(\|B\|_{L_1[0,T]})^a$$

for some $a > 0$.