Chapter 3

Stochastic Differential Equations.

3.1 Existence and Uniqueness.

One of the ways of constructing a Diffusion process is to solve the stochastic differential equation

$$dx(t) = \sigma(t, x(t)) \cdot d\beta(t) + b(t, x(t))dt \; ; \; x(0) = x_0 \tag{3.1}$$

where $x_0 \in R^d$ is either nonrandom or measurable with respect to $\mathcal{F}_0$. This is of course written as a stochastic integral equation

$$x(t) = x(0) + \int_0^t \sigma(s, x(s)) \cdot d\beta(s) + \int_0^t b(s, x(s))ds \tag{3.2}$$

If $\sigma(s, x)$ and $b(s, x)$ satisfy the following conditions

$$|\sigma(s, x)| \leq C(1 + |x|) \; ; \; |b(s, x)| \leq C(1 + |x|) \tag{3.3}$$

$$|\sigma(s, x) - \sigma(s, y)| \leq C|x - y| \; ; \; |b(s, x) - b(s, y)| \leq C|x - y| \tag{3.4}$$

by a Picard type iteration scheme one can prove existence and uniqueness.

Theorem 3.1. *Given σ, b that satisfy (3.3) and (3.4), for given x_0 which is $\mathcal{F}_0$ measurable, there is a unique solution $x(t)$ of (3.2), with in the class of progressively measurable almost surely continuous solutions.*

Proof. Define iteratively

$$x_0(t) \equiv x_0$$

$$x_n(t) = x_0 + \int_0^t \sigma(s, x_{n-1}(s)) \cdot d\beta(s) + \int_0^t b(s, x_{n-1}(s))ds \tag{3.5}$$

If we denote the difference $x_n(t) - x_{n-1}(t)$ by $z_n(t)$, then

$$\begin{aligned} z_{n+1}(t) = &\int_0^t [\sigma(s, x_n(s)) - \sigma(s, x_{n-1}(s))] \cdot d\beta(s) \\ &+ \int_0^t [b(s, x_n(s)) - b(s, x_{n-1}(s))] ds \end{aligned}$$

If we limit ourselves to a finite interval $0 \le t \le T$, then

$$E\big[\big|\int_0^t [\sigma(s, x_n(s)) - \sigma(s, x_{n-1}(s))] \cdot d\beta(s)\big|^2\big] \le CE\big[\int_0^t |z_n(s)|^2 ds\big]$$

and

$$E\big[\big|\int_0^t [b(s, x_n(s)) - b(s, x_{n-1}(s))] ds\big|^2\big] \le CTE\big[\int_0^t |z_n(s)|^2 ds\big]$$

Therefore

$$E\big[|z_{n+1}(t)|^2\big] \le C_T E\big[\int_0^t |z_n(s)|^2 ds\big]$$

With the help of Doob's inequality one can get

$$\Delta_{n+1}(t) = E\big[\sup_{0 \le s \le t} |z_{n+1}(s)|^2\big] \le C_T E\big[\int_0^t |z_n(t)|^2 dt\big] \le C_T \int_0^t \Delta_n(s) ds$$

By induction this yields

$$\Delta_n(t) \le A \frac{C_T^n t^n}{n!}$$

which is sufficient to prove the existence of an almost sure uniform limit $x(t)$ of $x_n(t)$ on bounded intervals $[0, T]$. The limit $x(t)$ is clearly a solution of (3.2). Uniqueness is essentially the same proof. For the difference $z(t)$ of two solutions one quickly establishes

$$E\big[|z(t)|^2\big] \le C_T E\big[\int_0^t |z(s)|^2 ds\big]$$

which suffices to prove that $z(t) = 0$. □

Once we have uniqueness one should thing of $x(t)$ as a map of x_0 and the Brownian increments $d\beta$ in the interval $[0, t]$. In particular $x(t)$ is a map of $x(s)$ and the Brownian increments over the interval $[s, t]$. Since $x(s)$ is $\mathcal{F}_s$ measurable, we can conclude that $x(t)$ is a Markov process with transition probability

$$p(s, x, t, A) = P[x(t; s, x) \in A]$$

where $x(t; s, x)$ is the solution of (3.2) for $t \ge s$, initialised to start with $x(s) = x$.

It is easy to see, by an application of Itô's lemma that

$$\begin{aligned}M(t) =& u(t,x(t)) - u(s,x(s)) - \int_s^t \big[\frac{\partial u}{\partial s}(s,x(s)) \\ &+ \frac{1}{2}\sum_{i,j} a_{i,j}(s,x(s))\frac{\partial^2 u}{\partial x_i \partial x_j}(s,x(s)) + \sum_i b_i(s,x(s))\frac{\partial u}{\partial x_i}(s,x(s))\big]ds\end{aligned}$$

is a martingale, where $a = \sigma\sigma^*$, i.e.

$$a_{i,j}(s,x) = \sum_k \sigma_{i,k}(s,x)\sigma_{k,j}(s,x)$$

The process $x(t)$ is then clearly the Diffusion process associated with

$$L_s = \frac{1}{2}\sum_{i,j} a_{i,j}(s,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(s,x)\frac{\partial}{\partial x_i}$$

3.2 Smooth dependence on parameters.

If σ and b depend smoothly on an additional parameter θ then we will show that the solution $x(t) = x(t,\theta)$ will depend smoothly on the parameter. The idea is to start with the solution

$$x(t,\theta) = x_0(\theta) + \int_0^t \sigma(s,x(s,\theta),\theta)\cdot d\beta(s) + \int_0^t b(s,x(s,\theta),\theta)ds$$

Differentiating with respect to θ, and denoting by Y the derivative, we get

$$\begin{aligned}Y(t,\theta) = y_0(\theta) &+ \int_0^t \big[\sigma_x(s,x(s,\theta),\theta)Y(s,\theta) + \sigma_\theta(s,x(s,\theta),\theta)\big]\cdot d\beta(s) \\ &+ \int_0^t \big[b_x(s,x(s,\theta),\theta)Y(s,\theta) + b_\theta(s,x(s,\theta),\theta)\big]ds\end{aligned}$$

We look at (x,Y) as an enlarged system satisfying

$$\begin{aligned}dx =& \sigma\cdot d\beta + b dt \\ dY =& [\sigma_x Y + \sigma_\theta]\cdot d\beta + [b_x Y + b_\theta]dt\end{aligned}$$

The solution Y can easily be shown to be the derivative $D_\theta x(t,\theta)$. This procedure can be repeated for higher derivatives. We therefore arrive at the following Theorem.

Theorem 3.2. *Let σ, b depend on θ in such a way that all derivatives of σ and b with respect to θ of order at most k, exist and are uniformly bounded. Then the random variables $x(t,\theta)$ have derivatives with respect to θ up to order k and we can get moment estimates of all orders for these derivatives.*

Proof. We can go through the iteration scheme in such manner that the approximations Y_n are the derivatives of x_n. Therefore the limit Y has to be the derivative of x. The moment estimates on the other hand depend only on general results, stated in Lemma 3.3 below, concerning solutions of stochastic differential equations. Successive differentiations produces for the highest derivative $Z = Z_k$ of order k a linear equation of the form

$$dZ(t) = \sigma_k(t,\omega))Z(t)d\beta(t) + b_k(t,\omega)Z(t)dt + c_k(t,\omega)d\beta(t) + e_k(t,\omega)dt \quad (3.6)$$

where σ_k and b_k involve derivatives of $\sigma(t,x)$ and $b(t,x)$ with respect x and are uniformly bounded progressively measurable functions. c_k, d_k on the other hand are polynomials in lower order derivatives with progressively measurable bounded coefficients. One gets estimates on the moments of $Z = Z_k$ by induction on k. $\square$

Lemma 3.3. *Let σ_k, b_k be uniformly bounded*

$$\sup_{\omega, 0\le t\le T} [\|\sigma_k(t,\omega))\| + \|b_k(t,\omega)\|] \le C(T) < \infty$$

and c_k, e_k satisfy moment estimates of the form

$$\sup_{0\le t\le T} E\big[\|c_k(t,\omega)\|^r + \|e_k(t,\omega)\|^r\big] \le C_r(T) < \infty$$

Then the solution $Z(t)$ of (3.6) has finite moments of all orders and

$$\sup_{0\le s\le T} E\big[\|Z(s)\|^r\big] \le \tilde{C}_r(T) < \infty$$

Proof. We consider $u_r(x) = (1 + \|x\|^2)^{\frac{r}{2}}$ and use Itô's formula. We apply Hölder's inequality to separate the terms with both Z and c_k or e_k.

$$E\big[u_r(Z(t))\big] \le E\big[u_r(Z(0))\big] + A_r \int_0^t E\big[u_r(Z(s))\big]ds + \int_0^t B_r(s)ds$$

which is sufficient to provide the estimates. Again Doob's inequality makes it possible to take the supremum inside the expectation with out any trouble. $\square$

Corollary 3.4. *If $x(t)$ is viewed as a function of the starting point x, then one can view x as the parameter and conclude that if the coefficients have bounded derivatives of all orders then the solution $x(t)$ is almost surely an infinitely differentiable function of its starting point.*

Remark 3.1. Since smoothness is a local property, if σ and b have at most linear growth, the solution exists for all time with out explosion, and then one can modify the coefficients outside a bounded domain with out changing much. This implies that with out uniform bounds on derivatives the solutions $x(t)$ will still depend smoothly on the initial point, but the derivatives may not have moment estimates.

Remark 3.2. This means one can view the solution $u(t,x)$ of the equation

$$du(t,x) = \sigma(u(t,x)) \cdot d\beta + b(u(t,x))dt;\ u(0,x) = x$$

as random flow $u(t) : R^d \to R^d$. The flow as we saw is almost surely smooth.

3.3 Itô and Stratonovich Integrals.

In the definition of the stochastic integral

$$\eta(t) = \int_0^t f(s)dx(s)$$

we approximated it by sums of the form

$$\sum_j f(t_{j-1})[x(t_j) - x(t_{j-1})]$$

always sticking the increments in the future. This allowed the integrands to be more or less arbitrary, so long as it was measurable with respect to the past. This meshed well with the theory of martingales and made estimation easier. Another alternative, symmetric with respect to past and future, is to use the approximation

$$\sum_j \frac{[f(t_{j-1}) + f(t_j)]}{2}[x(t_j) - x(t_{j-1})]$$

It is not clear when this limit exists. When it exists it is called the Stratonovich integral and is denoted by $\int f(s) \circ dx(s)$. If $f(s) = f(s, x(s))$, then the difference between the two integrals can be explicitly calculated.

$$\int_0^t f(s, x(s)) \circ dx(s) = \int_0^t f(s, x(s)) \cdot dx(s) + \frac{1}{2}\int_0^t a(s)ds$$

where

$$\int_0^t a(s)ds = \lim \sum_j [f(t_j, x(t_j)) - f(t_{j-1}, x(t_{j-1}))][x(t_j) - x(t_{j-1})]$$

If $x(t)$ is just Brownian motion in R^d, then $a(s) = (\nabla \cdot f)(s, x(s))$. More generally if

$$\lim \sum_j [x_i(t_j) - x_i(t_{j-1})][x_k(t_j) - x_k(t_{j-1})] = \int_0^t a_{i,k}(s)ds$$

then

$$a(s) = \sum_{i,k} f_{i,k}(s, x(s))a_{i,k}(s) = Tr[(Df)(s, x(s))a(s)]$$

Solutions of

$$dx(t) = \sigma(t, x(t)) \cdot d\beta(t) + b(t, x(t))dt$$

can be recast as solutions of

$$dx(t) = \sigma(t, x(t)) \circ d\beta(t) + \tilde{b}(t, x(t))dt$$

with b and $\tilde{b}$ related by

$$b_i(t, x) = \tilde{b}_i(t, x) + \frac{1}{2}\sum \sigma_{j,k}(t, x)\frac{\partial}{\partial x_j}\sigma_{i,k}(t, x)$$

To see the relevance of this, one can try to solve

$$dx(t) = \sigma(t, x(t)) \cdot d\beta_j(t) + b(t, x(t))dt$$

by approximating $\beta(t)$ by a piecewise linear approximation $\beta^{(n)}(t)$ with derivative $f^{(n)}(t)$. Then we will have just ODE's

$$\frac{dx^{(n)}(t)}{dt} = \sigma(t, x^{(n)}(t))f^{(n)}(t) + b(t, x^{(n)}(t))$$

where $f^{(n)}(\cdot)$ are piecewise constant. An elementary calculation shows that over an interval of constancy $[t, t + h]$,

$$\begin{aligned} x_i^{(n)}(t+h) &= x_i^{(n)}(t) + \sigma(t, x^{(n)}(t)) \cdot Z_h + b_i(t, x^{(n)}(t))h \\ &\quad + \frac{1}{2} < Z_h, c_i(t, x^{(n)}(t))Z_h > + o((Z_h)^2) \end{aligned}$$

where

$$c_i(t, x) = \sum \sigma_{j,k}(t, x) \frac{\partial}{\partial x_j} \sigma_{i,k}(t, x)$$

and Z_h is a Gaussian with mean 0 and variance hI while

$$\beta^{(n)}(t + h) = \beta_n(t) + Z_h$$

It is not hard to see that the limit of $x^{(n)}(\cdot)$ exists and the limit solves

$$dx(t) = \sigma(t, x(t)) \cdot d\beta(t) + b(t, x(t))dt + \frac{1}{2}c(t, x(t))dt$$

or

$$dx(t) = \sigma(t, x(t)) \circ d\beta(t) + b(t, x(t))dt$$

It is convenient to consider a vector field

$$X = \sum_i \sigma_i(x) \frac{\partial}{\partial x_j}$$

and its square

$$X^2 = \sum_{i,j} \sigma_i(x)\sigma_j(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_j c_j(x) \frac{\partial}{\partial x_j}$$

where

$$c_j(x) = \sum_i \sigma_i(x) \frac{\partial \sigma_j(x)}{\partial x_i}$$

Then the solution of

$$dx(t) = \sigma(t, x(t)) \circ d\beta(t) + b(t, x(t))dt = \sum X_i(t, x(t)) \circ d\beta_i(t) + Y(t, x(t))dt$$

is a Diffusion with generator

$$L_t = \frac{1}{2}\sum X_i(t)^2 + Y(t)$$

When we change variables the vector fields change like ordinary first order calculus and

$$\widehat{L}_t = \frac{1}{2}\sum \widehat{X}_i(t)^2 + \widehat{Y}(t)$$

and the Stratonovich solution

$$dx(t) = \sigma(t, x(t)) \circ d\beta(t) + b(t, x(t))dt$$

transforms like

$$dF(x(t)) = DF \cdot dx(t) = (DF)(x(t))[\sigma(t, x(t)) \circ d\beta(t) + b(t, x(t))dt]$$

The Itô corrections are made up by the difference between the two integrals.

Remark 3.3. Following up on remark (3.1), for each $t > 0$, the solution actually maps $R^d \to R^d$ as a diffeomorphism. To see this it is best to view this through Stratonovich equations. Take $t = 1$. If the forward flow is therough vector fileds $X_i(t), Y(t)$, the reverse flow is through $-X_i(1-t), Y(1-t)$ and the reversed noise is $\hat{\beta}(t) = \beta(1) - \beta(1-t)$. One can see by the piecewise linear approximations that these are actually inverses of each other.

Remark 3.4. One can perhaps consider more general solutions to the stochastic differential equation. On some $(\Omega, \mathcal{F}_t, P)$, one can try to define $x(t)$ and $\beta(t)$, both progressively measurable, that satisfy the relation

$$x(t) = x(0) + \int_0^t \sigma(s, x(s)) \cdot d\beta(s) + \int_0^t b(s, x(s))ds$$

$x(t, \omega)$ may not be measurable with respect to $\mathcal{G}_t$ the sub σ-field generated by $\beta(\cdot)$. It is not hard to show, assuming Lipshhitz conditions on σ and b, that the Picard iteration scheme produces a solution of the above equation which is a progressively measurable function of $x(0)$ and the Brownian increments and any other solution is equal to it.