Let us return to our model. We have a solution of Kolmogorov's forwrad equation

$$\frac{\partial f_N(t,\xi_1,\ldots,\xi_N)}{\partial t} = N^2 \mathcal{A}_N f_N$$

where f_N is the density with respect to the invariant measure give by $\exp[-\sum \phi(\xi_i)]$. We saw that the Dirichlet form satisfies.

$$\int \frac{\sum |(D_i - D_{i+1}) f_N|^2}{f_N} e^{-\sum \phi(\xi_i)} d\xi \le \frac{C}{N}$$

If we denote the marginal in a block of length ℓ around x by $f_{N,x\ell}(\xi_1,\ldots,\xi_\ell)$ and by $\bar{f}_{N<\ell}$ the average

$$\bar{f}_{N,\ell} = \frac{1}{N}\sum_x f_{N,x,\ell}$$

then the Dirichlet form

$$\int \frac{\sum_{1\le i\le \ell-1} |(D_{i+1} - D_i)\bar{f}_{N,\ell}|^2}{\bar{f}_{N,\ell}} e^{-\sum \phi(\xi_i)} d\xi \le \frac{C\ell}{N^2}$$

With a log Sobolev constant of ℓ^2, we can apply log Sobolev inequality on each hyperplane $\sum_i x_i = \ell a$. If we write $\bar{f}_{N,\ell} e^{-\sum \phi(\xi_i)} d\xi = \bar{G}_{N,\ell}(da)\bar{\nu}_{N,\ell}(a,d\xi)$ and similarly $e^{-\sum_i \phi(\xi_i)} = \bar{\phi}_{N,\ell}(a)\bar{\mu}_{N,\ell}(a,d\xi)$

$$\int H(\bar{\nu}_{N,\ell}(a,\cdot)|\bar{\mu}_{N,\ell}(a,\cdot))\bar{G}_{N,\ell}(da) \le c\frac{\ell^3}{N^2}$$

If we can prove a conditional version of Cramér's Large deviation result, i.e. with $\lambda = h'(a)$, and

$$\bar{F}(a) = \frac{1}{M(\lambda)}\int e^{\lambda\xi - \phi(\xi)} F(\xi) d\xi$$

$$P[|\frac{1}{n}\sum_i F(\xi_i) - \bar{F}(a)| \ge \delta | \frac{1}{n}\sum \xi_i = a] \le \exp[-nC(\delta)]$$

where $C(\delta) > 0$, then it is not hard to see that with $\ell = N\epsilon$,

$$\int_{|\frac{1}{n}\sum_i F(\xi_i) - \bar{F}(a)| \ge \delta} \bar{f}_{N,\ell}(\xi) d\xi \le c\frac{(N\epsilon)^3}{N^3\epsilon} = c\epsilon^2$$

This will allow replacing averages like

$$\frac{1}{N}\sum J(\frac{x}{N}) F(\xi)$$

with

$$\frac{1}{N}\sum J(\frac{x}{N})\bar{F}(\frac{1}{2N\epsilon}\sum_{y:|y-x|\le N\epsilon} \xi_y)$$

with a small ϵ. The conditional version of Cramér's theorem is not hard to prove.

Let us start with i.i.d.r.v's with density $g(\xi)$. Let $g_n(\xi)$ be the density of $\frac{1}{n}\sum \xi_i$. Then

$$g_n(a) = \exp[-nh(a) + o(n)]$$

is the density version of the Large deviation result. Let us take this for granted. We want to compute

$$\lim_{n\to\infty} \frac{1}{n} \log E[e^{\rho \sum_i F(\xi_i)} | \sum_i \xi_i = na] = \psi(\rho, a)$$

Suppose we define

$$g_\rho(\xi) = \frac{1}{T(\rho)} e^{\rho F(\xi)} g(\xi)$$

normalized to be a probability distribution. Then the distribution of $\frac{1}{n}\sum \xi$ under the new distribution is given by a density $g_{n,\rho}(a)$ that will satisfy

$$g_{n,\rho}(a) = \exp[-nh(\rho, a) + o(n)]$$

It is not hard to see that

$$\psi(\rho, a) = \log T(\rho) - h(\rho, a) + h(a)$$

and

$$h(\rho, a) = \sup_\theta [\theta a - \log M(\rho, \theta)]$$

with

$$M(\rho, \theta) = \frac{1}{T(\theta)} \int e^{\rho F(\xi) + \theta \xi} g(\xi) d\xi$$

Getting exponential error estimate is just the differentiability of $\psi(\rho, a)$ at $\rho = 0$ which is obvious. Moreover

$$\psi_\rho(0, a) = \frac{T'(0)}{T(0)} - h_\rho(0, a)$$

and

$$h_\rho(0, a) = -\frac{M_\rho(0, \theta)}{M(0, \theta)} = \frac{T'(0)}{T(0)} - \frac{1}{M(0, \theta)} \int e^{\theta \xi} F(\xi) g(\xi) d\xi$$

with $\theta = h'(a)$. Finally

$$\psi_\rho(0, a) = \frac{1}{M(0, \theta)} \int e^{\theta \xi} F(\xi) g(\xi) d\xi$$

with $\theta = h'(a)$. Finally let us prove the density version of Cramér's theorem.

We write with $g_\theta(\xi) = \frac{1}{M(\theta)} e^{\theta \xi} g(\xi)$ and $\theta = h'(\xi)$, so that

$$\int \xi g_\theta(\xi) d\xi = a$$

$$\int_{\bar{\xi}=a} g(\xi_1)\cdots g(\xi_n)d\xi = M(\theta)^n e^{-na}\int_{\bar{\xi}=a} g_\theta(\xi_1)\cdots g_\theta(\xi_n)d\xi$$
$$= M(\theta)^n e^{-na} g_{n,\theta}(a)$$

Since now a is the mean, the density version of CLT yields

$$g_{n,\theta}(a) \simeq \frac{1}{\sqrt{2\pi n C(a)}}$$

Density version of CLT. It is sufficient to prove with mean 0 and variance 1

$$\lim_{n\to infty} \frac{1}{2\pi}\int \widehat{g}(\frac{t}{\sqrt{n}})^n e^{-itx}dt \to \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

It is not hard to get a bound of the form

$$|\widehat{g}(t)| \leq (1+ct^2)^{-c}$$

and for $n >> 1$,

$$(1+c\frac{t^2}{n})^{-nc} \leq (1+c't^2)^{-1}$$