Lecture 11.

Scaling limits of Large Systems. Start with an example. Let $\{x_i(t)\}$ be a collection of processes such that

$$dx_i(t) = c[x_{i-1}(t) - 2x_i(t) + x_{i+1}(t)]dt + \gamma_i(t)$$

where $c > 0$ and $\gamma_i(t) = \beta_{i,i+1}(t) - \beta_{i-1,i}(t)$. This is an example of a system of interacting processes. They are associated with sites on $\mathbf{Z}$ with interaction between neighbors. They share their stuff with neighbors giving some thing proportional to the difference with some noise.

$$dz_{i.i+1}(t) = c(x_i(t) - x_{i+1}(t))dt + d\beta_{i,i+1}(t)$$

$$dx_i(t) = dz_{i-1,i}(t)\,dt - dz_{i,i+1}(t)\,dt$$

We want to look at averages over large blocks. More precisely

$$\sum_i J(\frac{i}{N})x_i(t) = \langle J, \xi_N(t)\rangle$$

where J is a smooth function with compact support.

$$d\langle J, \xi_N(t)\rangle = c\langle \Delta_N J, \xi_N(t)\rangle + \langle \nabla_N J, d\beta(t)\rangle$$

If we speed up time by N^2, then as $N \to \infty$, if we denote the limit of $\xi_N(t)$ by $\xi(t)$, then

$$d\langle J, \xi(t)\rangle = c\langle \Delta J, \xi(t)\rangle$$

because the noise is negligible. The variance of the noise is

$$\frac{1}{N^2}\sum_i [J(\frac{i+1}{n}) - J(\frac{i}{N})]^2 . N^2 \simeq \frac{1}{N} \to 0$$

Leading to the equation

$$d\xi(t) = \Delta\xi(t)dt$$

If initiall there is a weak law of large numbers of the form

$$\langle J, \xi_N(0)\rangle \to \int J(q)m_0(q)dq$$

then at time N^t we have

$$\langle J, \xi_N(t)\rangle \to \int J(q)m(t,q)dq$$

where m solves the heat equation

$$m_t = c\Delta m; m(0,q) = m_0(q)$$

Now the question is what happens when the equation

$$dx_i(t) = c[x_{i-1}(t) - 2x_i(t) + x_{i+1}(t)]dt + \gamma_i(t)$$

is replaced by

$$dx_i(t) = [c(x_{i-1}(t)) - 2c(x_i(t)) + c(x_{i+1}(t))]dt + \gamma_i(t)$$

where $c(x)$ is nonlinear but like cx with $c > 0$. We can get as far as

$$d\langle J, \xi_N(t)\rangle = \langle \Delta_N J, \eta_N(t)\rangle + \langle \nabla_N J, d\beta(t)\rangle$$

where

$$\langle J, \eta_N(t)\rangle = \frac{1}{N}\sum_i J(\frac{i}{N})c(x_i(t))$$

The equations do not close. One has to find an expression for $\eta(t)$ in

$$\lim_{N\to\infty} \langle J, \eta_N(t)\rangle = \langle J, \eta(t)\rangle$$

in terms of $m(t)$ that appears as

$$\lim_{N\to\infty} \langle J, \xi_N(t)\rangle = \int J(q)m(t,q)dq$$

The answer is once again the ergodic theorem. If $c(x) = \phi'(x)$ for some nice ϕ then the product measure

$$d\mu = \Pi e^{-\phi(x_i)}dx_i$$

is seen to be invariant. Since $c(x) + \lambda$ and $c(x)$ can not be distinguished,

$$d\mu_\lambda = \Pi e^{-\phi(x_i)+\lambda x_i}dx_i$$

is invariant as well. The idea is that if the local mean is m then locally the variables x_i are distributed like the product measure μ_λ with λ chosen such that

$$\frac{1}{k(\lambda)}\int e^{-\phi(x)+\lambda x}x dx = m$$

where

$$k(\lambda) = \int\int e^{-\phi(x)+\lambda x}dx$$

is the normalization constant. It is easy to see that

$$m(\lambda) = \frac{k'(\lambda)}{k(\lambda)}$$

is nondecreasing in λ (Jensen's inequality says that $\log k(\lambda)$ is convex in λ). It can be inverted and $\lambda = \lambda(m)$. It is easy to see integrating by parts, that

$$\frac{1}{k(\lambda)} \int c(x) e^{-\phi(x)+\lambda x} dx = \lambda$$

since $c(x) = -\phi'(x)$. This suggests that

$$\eta(t,q) = \lambda(m(t,q))$$

giving us in the limit a nonlinear heat equation

$$m_t = \Delta \lambda(m(t,q))$$

Proof: The proof needs some assumptions and simplifications. We shall work on $\mathbf{Z}_N$ the integers modulo N, rather than $\mathbf{Z}$. In the limit our space will be circle rather than the line. So we do not have an infinite system, but only a large system of size N, i.e a diffusion on R^N. But $x_1 + x_2 + \cdots + x_N$ is conserved. So we really have a one parameter family of diffussions on hyperplanes $\frac{1}{N}\sum_{i=1}^N x_i = m$. The density

$$\Phi_N(x_1, x_2, \ldots, x_N) = e^{-\sum_{i=1}^N \phi(x_i)}$$

is invariant, but not ergodic. The conditional distributions given the mean are ergodic. Let us start initially with a random configurations according to the density $f_N(x_1, x_2, \ldots, x_N)$, satisfying

$$\int_{R^N} \log \frac{f_N(x_1, x_2, \ldots, x_N)}{\Phi_N(x_1, x_2, \ldots, x_N)} f_N(x_1, x_2, \ldots, x_N) dx_1 \cdots dx_N \le CN$$

for some $C < \infty$, independent of N. Then the distribution at time $N^2 t$ has a density $f_N(t, x_1, x_2, \ldots, x_N)$ and the average density

$$\overline{f}_N^T(x_1, x_2, \ldots, x_N) = \frac{1}{T}\int_0^T f_N(t, x_1, x_2, \ldots, x_N) dt$$

satisfies certain properties. The last property is the one we need, which is

$$\limsup_{\epsilon \to 0} \limsup_{N \to \infty} \int \Big| \frac{1}{N} \sum_{i=1}^N J\Big(\frac{i}{N}\Big)[c(x_i) - \lambda(\overline{x}_{i,N\epsilon})] \Big| \overline{f}_N^T(x_1, x_2, \ldots, x_N) dx_1 dx_2 \cdots dx_N = 0$$

where

$$\overline{x}_{i,N\epsilon} = \frac{1}{N\epsilon} \sum_{j:|i-j| \le N\epsilon} x_j$$