Lecture 10

Multiscale Problems. Averaging

Consider the SDE

$$dx(t) = \sqrt{\epsilon}\sigma_1(y(t))d\beta_1(t) + \epsilon b_1(y(t))dt$$
$$dy(t) = \sigma_2(y(t))d\beta_2(t) + b_2(y(t))dt$$

where β_1, β_2 are independent Brownian motions and σ_2, b_2 are periodic functions in y of period 1. $x(t)$ changes slowly. We can therefore "freeze" x and let y evolve. y moves on the circle with generator

$$\frac{1}{2}\sigma_2^2(y)D_y^2 + b_2(y)D_y$$

and will have a unique invariant density on the circle, i.e a periodic solution of

$$\frac{1}{2}[\sigma_2^2(y)\phi(y)]_{yy} = [b_2(y)\phi(y)]_y$$

with $\int_0^1 \phi(y)dy = 1$. If we average

$$\bar{b}_1 = \int_0^1 b_1(y)\phi(y)dy$$

and

$$\bar{\sigma}_1^2 = \int_0^1 \sigma_1^2(y)\phi(y)dy$$

then the process $x(\frac{t}{\epsilon})$ will converge to the diffusion with generator

$$\frac{1}{2}\bar{\sigma}_1^2 D_x^2 + \bar{b}_1 D_x$$

The idea of the proof is to use martingales. If we denote by $x_\epsilon(t), y_\epsilon(t)$, the speeded up processes $x(\frac{t}{\epsilon}), y(\frac{t}{\epsilon})$, or better still denote by Q_ϵ, the measure corresponing to it then

$$f(x(t)) - \int_0^t [\frac{1}{2}\sigma_1^2(y(s))f''(x(s)) + b_1(y(s))f'(x(s))]ds$$

is a martingale. In particular the marginal Q_ϵ^1 of the x component alone is tight and let Q^1 be a limit point. We would like to show that

$$\int_0^t [\frac{1}{2}\sigma_1^2(y(s))f''(x(s)) + b_1(y(s))f'(x(s))]ds$$

can be replaced by

$$\int_0^t [\frac{1}{2}\bar{\sigma}_1^2 f''(x(s)) + \bar{b}_1 f'(x(s))]ds$$

Since the Q_ϵ^1 processes are tight the modulus of continuity of $x(t)$ is under control with probability nearly 1. We can there for pretend that $f'(x(s))$ and $f''(x(s))$ are piecewise constant. In which case, it is enough to prove

$$E^{Q_\epsilon}\left[|\int_{t_1}^{t_2} g(y(s))ds - (t_2 - t_1)\int g(y)\phi(y)dy|\right] \to 0$$

which is a consequence of the ergodic theorem.

This is made only slightly harder if we consider

$$dx(t) = \sqrt{\epsilon}\sigma_1(x(t), y(t))d\beta_1(t) + \epsilon b_1(x(t), y(t))dt$$
$$dy(t) = \sigma_2(y(t))d\beta_2(t) + b_2(y(t))dt$$

We are led to the diffusion with generator

$$\frac{1}{2}\overline{\sigma}_1^2(x)D_x^2 + \overline{b}_1(x)D_x$$

where

$$\overline{b}_1(x) = \int_0^1 b_1(x, y)\phi(y)dy$$

and

$$\overline{\sigma}_1^2(x) = \int_0^1 \sigma_1^2(x, y)\phi(y)dy$$

But it becomes much harder if we consider

$$dx(t) = \sqrt{\epsilon}\sigma_1(x(t), y(t))d\beta_1(t) + \epsilon b_1(x(t), y(t))dt$$
$$dy(t) = \sigma_2(x(t), y(t))d\beta_2(t) + b_2(x(t), y(t))dt$$

Now the y process is influenced by the x process and there is no real ergodic theorem for the y process. Instead there is a whole family of ergodic theorems with invariant densities $\phi(x, y)$ depnding on the value of x. The question is still the replacement of

$$\int_0^t f(x(s), y(s))ds$$

by

$$\int_0^t \overline{f}(x(s))ds$$

where

$$\overline{f}(x) = \int f(x, y)\phi(x, y)dy$$

Equivalently the problem is to show that if $\overline{f}(x) \equiv 0$, then $\int_0^t f(x(s), y(s))ds$ is negligible. If $\overline{f}$ is zero, then by Fredholm alternative the equation

$$\frac{1}{2}\sigma_2^2(x, y)u_{yy}(x, y) + b_2(x, y)u_y(x, y) = f(x, y)$$

has a solution. By making $\int u(x, y)\phi(x, y) = 0$, the solutions can be chosen to depend nicely on x. Let us suppose that we have a nice $u(x, y)$ solving the above equation. With respect to Q^ϵ, with generator

$$\mathcal{L}_x + \frac{1}{\epsilon}\mathcal{L}_y$$

$$\epsilon u(x(t),y(t)) - \epsilon u(x(0),y(0)) - \epsilon\int_0^t [(\mathcal{L}_x u)(x(s),y(s))) + \frac{1}{\epsilon}(\mathcal{L}_y u)(x(s),y(s))]ds$$

is a martingale. Everything is small here as $\epsilon \to 0$, except

$$\int_0^t f(x(s),y(s))ds$$

and so in the limit it is a continuous martingale of bounded variation and is 0. To actually prove it is not hard. Suppose

$$A(t) = \int_0^t b(s)ds + M(t); \quad A(0) = M(0) = 0$$

where M is a martingale, then

$$E[A^2(t)] = E[M^2(t)] + 2\int_0^t A(s)b(s)ds$$

If b is bounded and A is small then $M(t)$ and $\int_0^t b(s)ds$ are both small.