Lecture 6.

More on Localization:

Suppose on the space $C[[0,T],R^d],\mathcal{F}_t,P]$ we have a stopping time τ and a family $Q_{\tau(\omega),\omega}$ of measures on $C[[\tau(w),T],R^d]$ such that

$$P\big[\omega : Q_{\tau(\omega),\omega}[y(\cdot) : y(\tau(\omega)) = x(\tau(\omega),\omega)] = 1\big] = 1$$

i.e. with probability 1 the Q-paths start where the P-paths ended. The we can define a measure $\widehat{P}$ on $C[[0,T],R^d],\mathcal{F}_t]$ such that $P = \widehat{P}$ on $\mathcal{F}_\tau$ and the r.c.p.d of $\widehat{P}$ given $\mathcal{F}_\tau$ is $Q_{\tau(\omega),\omega}$ on $\mathcal{F}_T^{\tau(\omega)}$ for almost all ω w.r.t. P. It is routine to do it. For instance

$$\widehat{P}[x(t)\in A] = P[\{x(t)\in A\}\cap\{\tau\geq t\}] + \int_{\tau<t} Q_{\tau(\omega),\omega}[x(t)\in A]P(d\omega)$$

If $X(t)$ is a martingale with respect to $Q_{\tau(\omega),\omega}$ for $t\geq\tau(\omega)$ [for almost all ω with respect to P] and $x(\tau\wedge t)$ is a martingale with respect to P, then $X(t)$ is a martingale with respect to $\widehat{P}$. To prove it we write for $A\in\mathcal{F}_s$ and $t>s$,

$$\begin{aligned}
\int_A X(t)d\widehat{P} &= \int_{A\cap\{\tau(\omega)\geq t\}} X(t)d\widehat{P} + \int_{A\cap\{s\leq\tau(\omega)<t\}} X(t)d\widehat{P} + \int_{A\cap\{\tau(\omega)<s\}} X(t)d\widehat{P}\\
&= \int_{A\cap\{\tau(\omega)\geq t\}} X(t)d\widehat{P} + \int_{A\cap\{s\leq\tau(\omega)<t\}} X(\tau(\omega))d\widehat{P} + \int_{A\cap\{\tau(\omega)<s\}} X(s)d\widehat{P}\\
&= \int_{A\cap\{\tau(\omega)\geq t\}} X(t)dP + \int_{A\cap\{s\leq\tau(\omega)<t\}} X(\tau(\omega))dP + \int_{A\cap\{\tau(\omega)<s\}} X(s)d\widehat{P}\\
&= \int_{A\cap\{s\leq\tau(\omega)\}} X(\tau(\omega)\wedge t)dP + \int_{A\cap\{\tau(\omega)<s\}} X(s)d\widehat{P}\\
&= \int_{A\cap\{s\leq\tau(\omega)\}} X(\tau(\omega)\wedge s)dP + \int_{A\cap\{\tau(\omega)<s\}} X(s)d\widehat{P}\\
&= \int_{A\cap\{s\leq\tau(\omega)\}} X(s)d\widehat{P} + \int_{A\cap\{\tau(\omega)<s\}} X(s)d\widehat{P}\\
&= \int_A X(s)d\widehat{P}
\end{aligned}$$

We can piece martingales together and piece measures together. We can also take measures apart by conditioning.

Theorem. (from Harmonic analysis). Conside the solution

$$u(s,x) = \int_s^T g(s,x,t,y)f(t,y)dtdy$$

of

$$u_s + \frac{1}{2}\Delta u + f(s,x) = 0; \quad u(T,x) = 0$$

Then if $1 < p < \infty$,

$$\|D_{x_i} D_{x_j} u\|_p \leq c(p,d)\|f\|_p$$

If $p \geq p(d)$, then

$$\|u\|_\infty \leq C_p \|f\|_p$$

One can then solve

$$u_s + \frac{1}{2}\sum_{i,j}[\delta_{i,j} + \epsilon_{i,j}(s,x)]D_{x_i} D_{x_j} u + f(s,x) = 0; \quad u(T,x) = 0$$

by perturbation, if $\epsilon_{i,j}(s,x)$ are all uniformly small. The solution will be in $W_P^{1,2}$. The bound on u_∞ allows us to prove an apriori bound

$$E^P\Big[\int_s^T |f(t,x(t))|dt\Big] \leq C_p\|f\|_p$$

for any stochastic integral of the form

$$x(t) = x + \int e(\tau,\omega)d\beta(\tau)$$

with

$$|(e\,e^*)_{i,j} - \delta_{i,j}| \leq \epsilon$$

for samll enough ϵ.

Step 1.

$$(D_s + \frac{1}{2}\Delta + E)^{-1} = (D_s + \frac{1}{2}\Delta)^{-1}[I + E(D_s + \frac{1}{2}\Delta)^{-1}]^{-1}$$

If $\|E(D_s + \frac{1}{2}\Delta)^{-1}\| < 1$ from $L_p \to L_p$ the perturbation works.

Step 2. If $x(t)$ is a stochastic integral of a simple function then the bound is valid with some constant if $p \geq p(d)$. By Itô's formula, for

$$u(s,x) = \int_s^T g(s,x,t,y)f(t,y)dtdy$$

we get for any stochastic integral $x(t) = \int e d\beta$,

$$u(s,x) = E\Big[\int_0^T [f(t,x(t)) + c(t,x(t))]dt\Big]$$

or

$$E\Big[\int_0^T f(t,x(t))dt\Big] = u(s,x) - E\Big[\int_0^T c(t,x(t))dt\Big]$$

Taking the supremum over all f with $\|f\|_p \le 1$,

$$\sup_{f:\|f\|_p\le 1} |E\Big[\int_0^T f(t,x(t))dt\Big]| \le C_p + \frac{\epsilon d^2}{2} c(p,d) \sup_{f:\|f\|_p\le 1} |E\Big[\int_0^T f(t,x(t))dt\Big]|$$

If $\epsilon << 1$ we are done! With localization and Girsanov we have unique solutions to the martingale problem provided $\{a_{i,j}(t,x)\}$ are bounded, continuous and positive definite and $\{b_j(s,x)\}$ are bounded and measurable.

Estimates on Exit times and explosions. Let $(\Omega, \mathcal{F}_t, P, x(t,\omega), a(t,\omega), b(t,\omega)$ be a solution to a martingale problem starting from $x(0) = 0$ with $[a(t,\omega), b(t,\omega)]$ satisfying

$$\sum_{i,j} |a_{i,j}(t,\omega)| + \sum_j \|b_j(t,\omega)\|^2 \le A[\epsilon^2 + \|x(t)\|^2]$$

for some $\epsilon > 0$ and $A < \infty$. Then the function

$$u(x) = [\epsilon^2 + \|x\|^2]^k$$

satisfies

$$\frac{1}{2}\sum_{i,j} a_{i,j}(t,\omega) u_{x_i,x_j}(x(t)) + \sum_j b_j(t,\omega) u_{x_j}(x(t)) \le Ck^2 u(x)$$

where C depends only on A. In particular

$$e^{-Ck^2 t} u(x(t))$$

is a supermartingale. If τ_ℓ is the exit time from the ball of radius ℓ, we have

$$E[e^{-Ck^2\tau_\ell}] \le \frac{\epsilon^{2k}}{(\epsilon^2 + \ell^2)^k}$$

and

$$P[\tau_\ell \le t] \le \frac{\epsilon^{2k} e^{Ck^2 t}}{(\epsilon^2 + \ell^2)^k}$$

Fixing $\epsilon = 1$, $k = 1$ and letting $\ell \to \infty$ we see that quadratic bound on a and linear bound on b, is sufficient to prevent explosion. On the other hand if we fix ℓ and $\epsilon \to 0$, we can afford to pick k large and get an estimate of the form

$$P[\tau_\ell \le t] \le e^{-\frac{(\log \frac{\ell}{\epsilon})^2}{Ct}}$$

provoded $\ell > \epsilon$. This quantifies uniqueness under Lipschitz condition. More on this later.

Large Deviations.

Let P_ϵ the Wiener measure with variance $\epsilon\min(s,t)$ or $x(t)=\sqrt{\epsilon}\beta(t)$. On $C[[0,T],R^d$ $P_\epsilon\to\delta_0$ the delta function at the path $x(t)\equiv 0$. If A is a closed and $0\notin A$, $P_\epsilon(A)\to 0$. How fast?

Theorem. For A closed

$$\limsup_{\epsilon\to 0}\epsilon\log P_\epsilon[A]\le -\inf_{\substack{f(0)=0\\ f(\cdot)\in A}}\frac{1}{2}\int_0^T\|f'(t)\|^2dt=-I(A)$$

and for G open

$$\liminf_{\epsilon\to 0}\epsilon\log P_\epsilon[G]\ge -\inf_{\substack{f(0)=0\\ f(\cdot)\in G}}\frac{1}{2}\int_0^T\|f'(t)\|^2dt=-I(G)$$

Proof: Let us denote by $x_n(t)$ the piecewise linear version of $x(t)$ interpolated at $\{\frac{jT}{N}\}$. Then

$$U_n=\int_0^T\|x_n'(t)\|^2dt=\frac{n}{T}\sum_{j=1}^n[x(\frac{jT}{n})-x(\frac{(j-1)T}{N})]^2$$

is distributed like $\epsilon\chi^2$ with n degrees of freedom.

$$P_\epsilon[U_n\ge\ell]=\frac{1}{2^{\frac{n}{2}}\Gamma(n)}\int_{\frac{\ell}{\epsilon}}^\infty e^{-\frac{u}{2}}u^{\frac{n}{2}-1}du$$

Therefore for fixed n,ℓ

$$\limsup_{\epsilon\to 0}\epsilon\log P_\epsilon[U_n\ge\ell]\le-\frac{\ell}{2}$$

On the other hand for fixed δ

$$P_\epsilon[\sup_{t\in[0,T]}\|x_n(t)-x(t)\|\ge\delta]\le ne^{-\frac{n\delta^2}{2\epsilon T}}$$

Since

$$P_\epsilon[A]\le P_\epsilon[x_n(\cdot)\in A^\delta]+P_\epsilon[\|x_n(\cdot)-x(\cdot)\|\ge\delta]$$

we have

$$\limsup_{\epsilon\to 0}\epsilon\log P_\epsilon[A]\le\max\{-I(A^\delta),-\frac{n\delta^2}{2T}\}$$

Let $n\to\infty$ and then $\delta\to 0$. Here $A^\delta=\cup_{f\in A}B(f,\delta)$. For the lower bound let f be a smooth function with $f(0)=0$. It is enough to show that for such functions, for any $\delta>0$,

$$\liminf_{\epsilon\to 0}\epsilon\log P_\epsilon[B(f,\delta)]\ge-\frac{1}{2}\int_0^T\|f'(t)\|^2dt$$

By trnaslation forrmula for Browninan motion (Girsanov) in this context,

$$\begin{aligned} P_\epsilon[B(f,\delta)] &= \int_{B(0,\delta)} \exp[-\frac{1}{\epsilon}\int_0^T f'(t)d\beta(t) - \frac{1}{2\epsilon}\int \|f'(t)\|^2 dt]dP_\epsilon \\ &= e^{-\frac{1}{2\epsilon}\int_0^T \|f'(t)\|^2 dt} \int_{B(0,\delta)} \exp\big[-\frac{1}{\epsilon}\int_0^T f'(t)d\beta(t)\big]dP_\epsilon \\ &\geq e^{-\frac{1}{2\epsilon}\int_0^T \|f'(t)\|^2 dt} P_\epsilon[B(0,\delta)] \\ &= e^{-\frac{1}{2\epsilon}\int_0^T \|f'(t)\|^2 dt}[1+o(1)] \end{aligned}$$

by Jensen's inequlaity, the symmetry of Brownian motion and the convergence of P_ϵ to δ_0 as $\epsilon \to 0$.

Brownian motion with a drift.

Consider now the process corresponding to

$$\frac{\epsilon}{2}\Delta + < b(x), \nabla >$$

i.e solution of

$$x(t) = x + \int_0^t b(x(s))ds + y(t)$$

where $y(\cdot)$ has distribution P_ϵ. $Q_{x,\epsilon}$ satisfies a similar Large deviation theorem, with

$$I_{x,b}(A) = \inf_{\substack{f(0)=x \\ f(\cdot)\in A}} \frac{1}{2}\int_0^T \|f'(t) - b(f(t))\|^2 dt$$

Assuming that b is Lipschitz the proof is just the observation that the map $y(\cdot) \to x(\cdot)$ is a continupus map of $C[[0,T];R^d]$ into itself.

The exit problem.

Consider a domain G and a vector field b with a globally stable fixed point in G, to which all of G is attracted. If we start the process corresponding to

$$L_\epsilon = \frac{\epsilon}{2}\Delta + < b(x), \nabla >$$

from the unique equilibrium point, where and how will the path exit for small but positive ϵ? For $\epsilon = 0$, the path does not exit. So it is going to take a long time for the process corresponding to small ϵ. Think of it as swimming against the current to get out of G. If you are tired it is no use waiting to catch your breath, because you have to fight the current to stay where you are unless you are at the equlibrium. You can always coast down and rest. Eventually you make a trip with enough energy to make it all the way through. So the bahavior of the path consists of innumerable number of failed attempts where it goes some distance, only to be dragged back to the equlibrium. This will take a long time of order $\exp\frac{C}{\epsilon}$, then finally a quick getaway, making a beeline to the boundary in time that is only of order 1. The path that takes the particle out is the most efficient path. (Intelligent design?)