Lecture 3.

If we specify $D_{t,\omega}$ as a progressively mesurable map of $(\Omega \times [0,T], \mathcal{F}_t)$ into the space of infinitely divisible distributions, as well as an initial distribution for the starting point $x(0) = x$ then we would like to associate a measure P on the space Ω of paths on $[0,T]$.

Definition of "progressively measurable": In general we definitely want our maps from $\Omega \times [0,T]$ to any space (Y, Σ) to be non-anticipating, i.e. $y(t,\omega)$ from $\Omega \to Y$ to be measurable with respect to the σ-field $\mathcal{F}_t$ of events observable upto time t. But technically we need a little bit more. To do natural things like define $\eta(t) = \int_0^t f(y(s,\omega))ds$ and get them to be again non-anticipating functions one needs the joint measurability of y as a function of t and ω, which does not follow from measurability in ω alone. If the map is continuous in t for each ω then for every $t > 0$ the map $y : \Omega \times [0,t] \to Y$ is jointly measurable with respect to the product σ-field $\mathcal{F}_t \times \mathcal{B}_t$, where $\mathcal{F}_t$ is the σ-field of events observable upto time t and $\mathcal{B}_t$ is the Borel σ-field on the interval $[0,t]$. In general we have to assume this property, known as *progressive measurability.* With this, natural operations like integrals can be performed and result again in functions with the same property.

How should P be related to $D_{t,\omega}$?. The intuitive infinitesimal picture that we gave is hard to deal with in mathematically rigorous fashion. We turn instead to an essentially equivalent integral formulation with the help of martingales.

Let $f(x)$ be a smooth function on R. We can define for any infinitely divisible distribution or equivalently for any triplet $[b, \sigma^2, M]$ in the Levy-Khintchine representation the operator $\mathcal{A}$ acting on f by

$$(\mathcal{A}f)(x) = bf'(x) + \frac{\sigma^2}{2} f''(x) + \int [f(x+y) - f(x) - \frac{yf'(x)}{1+y^2}]M(dy)$$

This operator commutes with translations and it is not hard to evaluate

$$\mathcal{A}e^{i\xi x} = \psi(\xi)e^{i\xi x}$$

$\mathcal{A}$ is the infinitesimal generator of the semigroup $T_t = e^{t\mathcal{A}}$ which is convolution by μ_t with characteristic function $e^{t\psi(\xi)}$. In other words $\mathcal{A}$ is the infinitesimal generator

$$\mathcal{A} = \lim_{t \to 0} \frac{T_t - I}{t}$$

and

$$(T_t f)(x) = \int f(x+y)\mu_t(dy) = E[f(x(t))|x(0) = x]$$

where the expectation is with respect to the process with independent increments $x(t)$ which has distribution μ_t and characteristic function $e^{t\psi(\xi)}$.

If $D_{t,\omega}$ is given then we have $[b(t,\omega), \sigma^2(t,\omega), M_{t,\omega}(dy)]$. Assuming they depend reasonably on t, ω we have the operator

$$(\mathcal{A}_{t,\omega} f) = b(t,\omega) f'(x) + \frac{\sigma^2(t,\omega)}{2} f''(x) + \int [f(x+y) - f(x) - \frac{y f'(x)}{1+y^2}] M_{t,\omega}(dy)$$

and for each f, $u(t,\omega,x) = (\mathcal{A}_{t,\omega} f)(x)$ is a progressively meausrable map into the space of smooth functions on R. The natural boundedness assumption is to assume that

$$\sup_{\substack{0 \le t \le T \\ \omega \in \Omega}} [|b(t,\omega)| + \sigma^2(t,\omega) + \int \frac{y^2}{1+y^2} M_{t,\omega}(dy)] < \infty$$

One could in addition assume that $u(t,\omega,x)$ is a continuous functional of its variables. Then the realtionship between P and $D_{t,\omega}$ can be expressed as follows.

We say that P is a solution to the "Martingale Problem" corresponding to $D_{t,\omega}$ or equivalently $[b(t,\omega), \sigma^2(t,\omega), M_{t,\omega}(dy)]$ starting from x at time 0, if for every smooth f

$$Z_f(t) = f(x(t)) - f(x(0)) - \int_0^t (\mathcal{A}_{s,\omega} f)(x(s)) ds = f(x(t)) - f(x(0)) - \int_0^t u(s,\omega,x(s)) ds$$

is a martingale with respect to $(\Omega, \mathcal{F}_t, P)$ and $P[x(0) = x] = 1$. Even if it is assumed only for functions f that depend on x alone, it extends automatically to functions f that depend smoothly on x and t. If we freeze t and apply $\mathcal{A}_{t,\omega}$ with

$$u(t,x,\omega) = \mathcal{A}_{t,\omega} f(t,\cdot)(x)$$

$$Z_f(t) = f(t,x(t)) - f(0,x(0)) - \int_0^t \big[\frac{\partial f}{\partial s}(s,x(s)) + u(s,\omega,x(s))\big] ds$$

is a martingale with respect to $(\Omega, \mathcal{F}_t, P)$. This is not hard to prove. We need to show $E[Z_f(t) - Z_f(s) | \mathcal{F}_s] = 0$. Let us show that

$$E^P[Z_f(t)] = 0$$

The conditional argument is identical. Let $V(s,\tau,\omega,x) = (\mathcal{A}_{\tau,\omega} f(s,\cdot))(x)$

$$\begin{aligned}
E^P[Z_f(t)] &= E^P\big[f(t,x(t)) - f(0,x(0)) - \int_0^t [f_s(s,x(s)) + V(s,s,\omega,x(s)) ds]\big] \\
&= E^P\big[f(t,x(t)) - f(t,x(0)) + f(t,x(0)) - f(0,x(0)) \\
&\qquad - \int_0^t [f_s(s,x(s)) + V(s,s,\omega,x(s)) ds]\big] \\
&= E^P\big[\int_0^t V(t,s,\omega,x(s)) ds + \int_0^t f_s(s,x(0)) ds \\
&\qquad - \int_0^t [f_s(s,x(s)) + V(s,s,\omega,x(s)) ds]\big] \\
&= E^P\big[\int_0^t ds \int_s^t V_\tau(\tau,s,\omega,x(s))) d\tau - \int_0^t ds \int_0^s V(s,\tau,\omega,x(\tau)) d\tau\big] \\
&= 0
\end{aligned}$$

If we use the fact that for any martingale $M(t)$ and a function of bounded variation $A(t)$ the quantity

$$M(t)A(t) - \int_0^t M(s)dA(s)$$

is again a martingale we can go, back and forth, between adidtive martingales

$$Z_f(t) = e^{f(t,x(t))} - e^{f(0,x(0))} - \int_0^t [(\frac{\partial}{\partial s} + \mathcal{A}_{s,\omega})e^f](s, x(s))ds$$

and multiplicative martingales

$$Y_f(t) = \exp\left[f(x(t)) - f(x(0)) - \int_0^t [e^{-f}((\frac{\partial}{\partial s} + \mathcal{A}_{s,\omega})e^f)(s, x(s))]ds\right]$$

This requires the choice of $M(t) = Z_f(t)$ and

$$A(t) = \exp\left[-\int_0^t [e^{-f}((\frac{\partial}{\partial s} + \mathcal{A}_{s,\omega})e^f)(s, x(s))]ds\right]$$

and a similar choice of $M(t) = Y_f(t)$ and

$$A(t) = \exp\left[\int_0^t [e^{-f}((\frac{\partial}{\partial s} + \mathcal{A}_{s,\omega})e^f)(s, x(s))]ds\right]$$

to go back.

Let us look at the case $M \equiv 0$. Then

$$e^{-f}((\frac{\partial}{\partial s} + \mathcal{A}_{s,\omega})e^f)(s,x) = f_s(s,x) + b(s,\omega)f_x(s,x) + \frac{\sigma^2(s,\omega)}{2}f_{xx}(s,x) + \frac{\sigma^2(s,\omega)}{2}|f_x(s,x)|^2$$

Let us take $f(x) = \theta x$. We get

$$Y_\theta(t) = \exp\left[\theta[x(t) - x(0) - \int_0^t b(s,\omega)ds] - \frac{\theta^2}{2}\int_0^t \sigma^2(s,\omega)ds\right]$$

We need to justify this because θx is unbounded. We can truncate and pass to the limit. By Fatou's lemma we will only get that $Y_f(t)$ is a super-martingale. This is enough to get a bound

$$E^P[Y_\theta(t)] \leq 1$$

Since σ^2 is bounded by C,

$$E^P\left[\exp\left[\theta[x(t) - x(0) - \int_0^t b(s,\omega)ds\right]\right] \leq e^{\frac{Ct\theta^2}{2}}$$

By Tchebechev's inequality, this will give a Gaussian type bound on

$$P[|x(t) - x(0) - \int_0^t b(s,\omega)ds| \geq \ell] \leq Ae^{-\frac{\ell^2}{2Ct}}$$

which can then be used to get uniform integrabilty and actually show that $Y_\theta(t)$ is a martingale.

Remark. If we just knew that $Y_\theta(t)$ are martingales, it is easy to see that one can continue analytically and replace θ by $i\theta$. We can then go from $Y_{i\theta}(t)$ to $Z_{i\theta}(t)$. By Fourier synthesis it is easy to pass from $\{e^{i\theta x} : \theta \in R\}$ to smooth functions f. In terms of $Y_\theta(t)$ it is easy to see that $y(t) = x(t) - \int_0^t b(s,\omega)ds$ corresponds to $[0, \sigma^2(t,\omega)]$.

Since we have uniform Gaussian estimates on any $y(t)$ corresponding to $[0, \sigma^2(t,\omega)]$ with a bounded σ^2 of the from

$$E\big[e^{\frac{\theta}{\sqrt{t}}[y(t)-y(0)]}\big] \leq e^{\frac{C\theta^2}{2}}$$

one can get estimates of the form

$$E[(y(t) - y(0))^{2n}] \leq c_n t^n$$

or

$$E[(y(t) - y(s))^{2n}] \leq c_n |t-s|^n$$

proving, by Kolmogorov's theorem tightness for any family $\{P_\alpha\}$ that corresponds to $[0, \sigma_\alpha^2(t,\omega)]$ provided there is a uniform bound on $\{\sigma_\alpha^2(t,\omega)\}$ as well as the starting points $\{x_\alpha\}$. Adding "b" causes no problem because $\{\int_0^t b_\alpha(s,\omega)ds\}$ are uniformly Lipshitz if we have a uniform bound on $\{b_\alpha(t,\omega)\}$.

We will use this to prove existence of P for given b, σ^2. Let us do this for the Markovian case where $b(t,\omega), \sigma(t,\omega)$ are given by $b(t, x(t))$ and $\sigma(t, x(t))$. We assume that b, σ are bounded continuous functions of t, x. If b and σ are constants then $bt + \sigma\beta(t)$ where $\beta(t)$ is Brownian motion will do it. We split the interval $[0, T]$ into subintervals of length h and run $x(t) = b(0,x)t + \sigma(0,x)\beta(t)$ upto time h. Then conditionally in the interval $[h, 2h]$ we run $x(t) = x(h) + b(h, x(h))(t-h) + \sigma(h, x(h))[\beta(t) - \beta(h)]$. The role of $\beta(t) - \beta(h)$ is to give us a Brownian motion independent of the past. We continue like this till time t. It is not hard to check, by induction if you want to be meticulous, that we now have a process P_h that is a solution to the martingale problem corresponding to $[b_h(t,\omega), \sigma_h^2(t,\omega)]$, where

$$b_h(t,\omega) = b(\pi_h(t), x(\pi_h(t)); \quad \sigma_h^2(t,\omega) = \sigma^2(\pi_h(t), x(\pi_h(t))$$

and $\pi_h(t) = \sup\{s : s \leq t, s = kh\}$ the last time of updating. Since the starting points are same and $\{b_h, \sigma^h\}$ are uniformly bounded P_h is tight. It is not hard to see that any limit point P of $\{P_h\}$ as $h \to 0$ will work.

$$E^{P_h}[Z_f^h(t) - Z_f^h(s)|\mathcal{F}_s] = 0$$

$P_h \to P$ and $Z_f^h \to Z_f$. One can easily pass to the limit if it were not for the conditioning. But all the conditioning is the relation

$$E^{P_h}\big[[Z_f^h(t) - Z_f^h(s)]G(\omega)\big] = 0$$

For any $\mathcal{F}_s$ measurable bounded function. It is enough to check for all bounded continuous functions G and in that case we can pass to the limit under weak convergence of P_h to P.

Uniquenss, Markov property etc. Suppose we can solve the PDE

$$u_s + b(s,x)u_x(s,x) + \frac{\sigma^2(s,x)}{2}u_{xx}(s,x) = 0; \quad u(t,x) = f(x)$$

for $0 \le s \le t$, and for any $t \le T$ for a large class of functions f and get a smooth solution. Then for any solution P of the martingale problem corresponding to $b(t,x), \sigma^2(t,x)$ that starts from x at time s

$$u(s,x) = E^P[f(x(t)]$$

Another way of saying it is that if we define by $C_{s,x}$ the set of all solutions to the martingale problem corresponding to $a(\cdot,\cdot)$ and $\sigma^2(\cdot,\cdot)$ then for any solution for any $P \in C_{s,x}$, $E^P[f(x(t))]$ is gineby $u(t,x)$ where u is a solution of the PDE above. If the solution to the PDE exists for sufficiently many f, then the distribution of of $x(t)$ under any $P \in C_{s.x}$ is dtermined and is given by $P(s,x,t,dy)$ that satisfies

$$u(s,x) = \int f(y)p(s,x,t,dy)$$

for all $0 \le s < t \le T$. One can alternatively solve

$$u_s(s,x) + b(s,x)u_x(s,x) + \frac{\sigma^2(s,x)}{2}u_{xx}(s,x) + f(s,x) = 0 \quad \text{in} \quad [0,T]\times R; u(T,x) = 0$$

Then for any $P \in C_{s,x}$ we would get

$$u(s,x) = E^P[\int_s^T f(t,x(t))dt]$$

and that would determine $P[x(t) \in A]$ as well. To dtermine P completely as well as to prove the Markov property it suffiecs to show that if $P \in C_{s,x}$ ans $s < t < T$ then the regular conditional probability distribution $Q_{t,\omega}$ of P given $\mathcal{F}_t$ is in $C_{t,x(t)}$ for almost all ω with respect to P. This amounts to proving

$$E^{Q_{t,\omega}}[Z_f(t') - Z_f(t)|\mathcal{F}_{t'}] = 0 \; a.e \; Q_{t,\omega}, \; a.e. \; P$$

In other words for $A \in \mathcal{F}_t, B \in \mathcal{F}_{t'}$

$$\int_A [\int_B [Z_f(t') - Z_f(t)]dQ_{t,\omega}]dP = 0$$

But this equals

$$\int_{A\cap B}[Z_f(t') - Z_f(t)]dP = 0$$

because $A \cap B \in \mathcal{F}_{t'}$. There are problems with sets of measure zero that now depend on f, A, B, t, t'. But countable choices can be made to get one null set and then the result extended by continuity.

By this argument if the PDE has enough solutions then $P \in C_{s,x}$ is Markov with transition probability $p(s, x, t, dy)$ and is unique. The same argument applies to stopping times by Doob's theorem on stopping times for Martingales and we deduce the strong Markov property as well. Once we know $\tau(\omega)$ and $x(\tau(\omega)), \omega)$, the process is the unique probability measure $P_{\tau,x(\tau)} \in C_{\tau,x(\tau)}$

Lecture 4.

One could in a similar fashion consider Markov processes with jumps determined by $b(t, x), \sigma^2(t, x), M(t, x, dy)$. and the corresponding operator

$$(\mathcal{A}_s f)(x) = b(s, x)f_x + \frac{\sigma^2(s, x)}{2} f_{xx} + \int [f(x + y) - f(x) - \frac{y f_x(x)}{1 + y^2}] M(s, x, dy)$$

Solutions to the Martingale problem will be determined in exactly similar manner by solving instead of the PDE, the integro diffrential equation

$$u_s(s, x) + (\mathcal{A}_s u(s, \cdot))(x) = 0; \quad u(t, x) = f(x)$$

or

$$u_s(s, x) + (\mathcal{A}_s u(s, \cdot))(x) + f(s, x) = 0; \quad u(t, x) = 0$$

for sufficiently many functions f. The martingales involved as well as the process itself are only right continuous and have left limits, but may have jumps. Otherwise the theory is very similar. For proving existence one works in the Skorohod space $D[0, T]$ instead of $C[0, T]$ which is the natural space for diffusions, i.e. Markov processes with out jumps.

In the homogeneous or time independent case, we have the operator

$$(\mathcal{A}f)(x) = b(x)f_x + \frac{\sigma^2(x)}{2} f_{xx} + \int [f(x + y) - f(x) - \frac{y f_x(x)}{1 + y^2}] M(x, dy)$$

In order to construct the semigroup T_t with generator $\mathcal{A}$ one can, in the semi group theory solve for the resolvent

$$R_\lambda = (\lambda I - \mathcal{A})^{-1} = \int_0^\infty e^{-\lambda t} T_t \, dt$$

and reconstruct T_t from R_λ as

$$T_t = \lim_{\lambda \to \infty} e^{t\lambda(R_\lambda - I)}$$

But one can use martingales instead of semigroup theory and if the equation

$$\lambda u_\lambda(x) - (\mathcal{A}u_\lambda)(x) = f(x)$$

has a solution, then

$$w(s, x) = e^{-\lambda s}u(x)$$

solves

$$w_s + (\mathcal{A}w(s,\cdot) + e^{-\lambda s}f(x) = 0$$

Therefore with respect to any $P \in C_{0,x}$,

$$w(t, x(t)) + \int_0^t e^{-\lambda s}f(x(s))ds$$

is a Martingale. Equating expectations at 0 and ∞, since $w(t, x) \to 0$ as $t \to \infty$, we get

$$u\lambda(x) = w(0, x) = E^P[\int_0^\infty e^{-\lambda s}f(x(s))ds$$

From the uniqueness theorem for Laplace transforms we determine

$$E^P[f(x(t)] = \int f(y)q(t, x, dy).$$

The rest of proceeds like the inhomogeneous case. The transition probabilities $p(s, x, t, dy)$ are given by $q(t - s, x, dy)$.

Stochastic Integrals, Itô's formula. Although we have seen examples of this, it is useful to introduce the following general concept. We have a probability space $(\Omega, \mathcal{F}, P)$ and an increasing family $\mathcal{F}_t$ of sub-σ-fields. One can assume that $\mathcal{F}$ is generated by $\cup_t\mathcal{F}_t$. Limiting ourselves to the continuous case given $x(t, \omega)$, $b(t, \omega)$ and $\sigma^2(t, \omega)$ that are progressively measurable,we say that $x(\cdot,\cdot)$ is an Itô process corresponding to $b(t, \omega), \sigma^2(t, \omega)$ on $(\Omega, \mathcal{F}_t, P)$, if $x(t, \omega)$ is almost surely continuous and $Z_f(t)$ are martingales with respect to $(\Omega, \mathcal{F}_t, P)$. We denote this in symbols by $x(\cdot) \in \mathcal{I}[b(\cdot,\cdot), \sigma^2(\cdot,\cdot)]$. We know for instance that if $x(\cdot) \in \mathcal{I}[b(\cdot,\cdot), \sigma^2(\cdot,\cdot)]$ and $y(t) = \int_0^t b(s, \omega)ds$ then $y(\cdot) \in \mathcal{I}[0, \sigma^2(\cdot,\cdot)]$. If we want to define stochastic integrals with respect to $x(\cdot)$ since $x(t) - y(t)$ is of bounded variation we can always assume that $b = 0$.

Simple functions. A simple function relative to a partition $0 = t_0 < t_1 < \cdots < t_n = T$ is a bounded function $c(s, \omega)$ which is equal to $c(t_{j-1}, \omega)$ in the interval $[t_{j-1}, t_j]$ with $c(t_{j-1}, \omega)$ being $\mathcal{F}_{t_{j-1}}$ measurable. One can define the stocahs tic integral $\xi(t, \omega)$ by

$$\begin{aligned}\xi(t) &= \int_0^t c(s)dx(s)\\ &= \sum_{i=1}^{j-1} c(t_{i-1}, \omega)[x(t_i) - x(t_{i-1})] + c(t_{j-1}, \omega)[x(t) - x(t_{j-1})]\end{aligned}$$

Clearly $\xi(t)$ is continuous and it is not hard to check that it is an Itô process corresponding to $[0, c^2(s,\omega)\sigma^2(s,\omega)]$. To see this we need to note that since

$$E^P[e^{\theta(x(t_j)-x(t_{j-1}))} - \frac{\theta^2}{2}\int_{t_{j-1}}^{t_j} \sigma^2(s,\omega)ds|\mathcal{F}_{t_{j-1}}] = 1$$

for all θ, we can take for θ a function $\theta\, c(t_{j-1},\omega)$ which is $\mathcal{F}_{t_{j-1}}$ measurable. It is now easy to check that $\xi(t)$ is a martingale and

$$E^P[\xi(t)] = 0; E^P[\xi^2(t)] = E^P[\int_0^t |c(s,\omega)|^2 ds]$$

Bounded progressively mesuarble integrands. If c is bounded and progressively measurable it can be approximated by a sequence c_n of simple functions such that

$$\lim_{n\to\infty} E^P[\int_0^T |c_n(s,\omega) - c(s,\omega)|^2 ds] = 0$$

For $h > 0$ one defines

$$c_h(s,\omega) = \frac{1}{h}\int_{s-h}^{s} c(s,\omega)ds$$

if $s > h$ and 0 otherwise. Then c_h are uniformly bounded and $\int_0^T |c_h(s,\omega) - c(s,\omega)|^2 ds \to 0$ a.e P. Therefore

$$E^P[\int_0^T |c_h(s,\omega) - c(s,\omega)|^2 ds] \to 0$$

as $h \to 0$. Now c_h is continuous in s and can be easily approximated by defining $c_h, n(s,\omega)$ to be $c_h(t_{j-1},\omega)$ on $[t_{j-1}, t_j]$. We approximate c by c_h and c_h by $c_{h,n}$. Now if $E^P[\int_0^T |c_n(s,\omega) - c(s,\omega)|^2 ds \to 0$, then by Doob's inequality

$$\lim_{n,m\to\infty} E^P[\sup_{0\le t\le T} |\xi_n(t) - \xi_m(t)|^2 dt] = 0$$

By taking a subsequence if needed there is an almost surely uniform limit of $\xi_n(t) = \int_0^t c_n(s,\omega)dx(s)$ and we define it to be $\xi(t) = \int_0^t c(s,\omega)dx(s)$. It is not hard to check that $\xi(t) \in \mathcal{I}[0, c^2(s,\omega)\sigma^2(s,\omega)]$.

Further Extensions. One can define $\xi(t) = \int_0^t c(s,\omega)dx(s)$ for unbounded $c(s,\omega)$ provided $E^P[\int_0^T |c(s,\omega)|^2 ds] < \infty$. Since $\sigma^2(s,\omega)$ is bounded a simple truncation gives it as a square integrable limit of bounded approximating ones obtaine by truncation. All the stochastic integrals are almost surely contnouos square integrable martingales with

$$E^P[|\xi(t)|^2] = E^P[\int_0^t |c(s,\omega)^2\sigma^2(s,\omega)ds]$$

One gets the exponetial martingales only if $c(s,\omega)\sigma(s,\omega)$ is bounded. The stochstic integrals are still Itô processes but correspond to unbounded parameters and can be defined by martingales $Z_f(t)$.

Multi-dimensional versions. The stochastic process is R^n valued. The defiining parameters are $\tilde{b}$ with components b_i, $1 \leq i \leq n$, a positive semi-definite symmetric matrix $\{a_{i,j}(s,\omega)\}$ and possibly a levy-measuure $M(s,\omega,dy)$ on $R^n/backslash\{0\}$. The operator $\mathcal{A}_{s,\omega}$ is given by

$$(\mathcal{A}_{s,\omega}f) = \sum b_i(s,\omega)f_{x_i}(x) + \frac{1}{2}\sum a_{i,j}(s,\omega)f_{x_ix_j}(x) \\ + \int [f(x+y) - f(x) - \frac{< y, (\nabla f)(x)}{1+\|y\|^2} M(s,\omega,dy)$$

The exponential martingales are

$$\exp\left[< \theta, x(t) - x(0) - \int_0^t < \theta, b(s,\omega) > ds - \frac{1}{2}\int_0^t < \theta, a(s,\omega)\theta > ds \right.$$

The PDE, in the Markov case is

$$u_t + < b(s,x), \nabla u(s,x) > + \frac{1}{2}\sum a_{i,j}(s,x)u_{x_i,x_j}(s,x) = 0, u(t,x) = f(x).$$

The stochastic integrals take the form

$$y(t) = \int_0^t c(s,\omega)\cdot dx(s)$$

where c is an $m \times n$ matrix. Then y takes values in R^m. The definitions are simple enough. We do it componentwise.

$$y_i(t) = \sum_j \int_0^t c_{i,j}(s,\omega)dx_j(s)$$

If $x(t) \in \mathcal{I}[b,a]$ then $y(t) \in \mathcal{I}[cb, cac^*]$.

Itô's formula. Let $f(t,x)$ be a smooth function and $x(t) \in \mathcal{I}[a(s,\omega),\sigma^2(s,\omega)]$ with bounded a and σ^2. We define $y(t) = f(t,x(t))$ and wish to consider the pair $x(t), y(t)$ as a two dimensional Itô process. Its chracteristic paramters will be $[\tilde{a}, \tilde{C}]$ a two vector $\tilde{a}$ and a 2 positive semidefinite matrix $\tilde{C}$. We can compute them by calculating

$$\frac{\partial F(x, f(x,t))}{\partial t} = F_2 f_t$$

and

$$F_t + \mathcal{A}F = F_2 f_t + b(s,\omega)[F_1 + F_2 f_x] + \frac{\sigma^2(s,x)}{2}[F_{11} + 2F_{1,2}f_x + F_{22}f_x^2 + F_2 f_{xx}]$$

One can read off from this

$$\tilde{b} = \{b(s,\omega)f_s(s,x(s)) + b(s,\omega)f_x(s,x(s)) + \frac{\sigma^2(s,\omega)}{2} f_{xx}\}$$

and

$$C = \begin{pmatrix} \sigma^2(s,\omega) & \sigma^2(s,\omega)f_x(s,x(s)) \\ \sigma^2(s,\omega)f_x(s,x(s)) & \sigma^2(s,\omega)f_x^2(s,x(s)) \end{pmatrix}$$

If this is not clear one can do it with the exponential martingale that makes it more transparent.

$$\exp[\lambda_1 x(t) + \lambda_2 f(t,x(t)) - \int_0^t G(\lambda_1,\lambda_2,s,\omega)ds]$$

is a martingale provided

$$G(\lambda_1,\lambda_2,s,\omega) = \exp[-\lambda_1 x(t) - \lambda_2 f(t,x(t))]\omega)\frac{\partial}{\partial x} + \frac{\sigma^2(s,\omega)}{2}\partial^2\partial x^2]\exp[\lambda_1 x(t) + \lambda_2 f(t,x(t))]da_1 x(t) + \lambda_2 f($$

G will be a secon degree polynomial in λ_1, λ_2 and the coefficients of the linear and quadratic terms determine $\tilde{b}, C$. Now we can define the stochastic integral

$$\eta(t) = \int_0^t < \{c_1(s,\omega), c_2(s,\omega)\}, \{dx(s), dy(s)\} >= \int_0^t c_1(s,\omega)dx(s) + \int_0^t c_2(s,\omega)dy(s)$$

and its parameters are

$$\mathbf{b} = c_1(s,\omega)b(s,\omega) + c_2(s,\omega)b(s,\omega)f_x(s,x(s)) + \frac{\sigma^2(s,\omega)}{2} f_{xx}(s,x(s))$$

and

$$\mathbf{a} = c_1(s,\omega)^2\sigma^2(s,\omega) + 2c_1(s,\omega)c_2(s,\omega)\sigma^2(s,\omega)f_x(s,x(s)) + c_2(s,\omega)^2\sigma^2(s,\omega)f_x^2(s,x(s))$$

If we choose $c_1(s,\omega) = -f_x(s,x(s))$ and $c_2 = 1$, then $\mathbf{a} = 0$ and $\mathbf{b} = \frac{\sigma^2(s,\omega)}{2} f_{xx}(s,x(s))$. In particular if

$$z(t) = y(t) - y(0) - \int_0^t [f_s(s,x(s))ds + f_x(s,x(s))dx(s) + \frac{\sigma^2(s,\omega)}{2} f_{xx}(s,x(s))ds$$

then $z(t) \in \mathcal{I}[0,0]$ and is in fact zero.