Probability/ Limit Theorems

Final Examination

Due before Dec 19

Q1. For each n, $\{X_{n,j}\}; j = 1, 2 \ldots, n$ are n mutually independent random variables taking values 0 or 1 with probabilities $1 - p_{n,j}$ and $p_{n,j}$ respectively. i.e

$$P[X_{n,j} = 1] = p_{n,j} \quad \text{and} \quad P[x_{n,j} = 0] = 1 - p_{n,j}$$

If

$$\lim_{n\to\infty} \sup_j p_{n,j} = 0,$$

then show that any limiting distribution of $S_n = X_{n,1} + X_{n,2} + \cdots + X_{n,n}$ is Poisson and the limit exists if and only if

$$\lambda = \lim_{n\to\infty} [p_{n,1} + p_{n,2} + \cdots + p_{n,n}]$$

exists, in which case the limit is Poisson with parameter λ.

Q2. Is the exponential distribution with density

$$f(x) = e^{-x} \text{ if } x \geq 0 \quad \text{and } 0 \text{ otherwise}$$

infinitely divisible? If it is, what is its Levy-Khintchine representation? How about the two sided exponential $f(x) = \frac{1}{2}e^{-|x|}$?

Q3. Let $f(x)$ be an integrable function on $[0, 1]$ with respect to the Lebesgue measure. For each n and $j = 0, 1, \ldots, 2^n - 1$ define for $j2^{-n} \leq x \leq (j + 1)2^{-n}$

$$f_n(x) = 2^n \int_{j2^{-n}}^{(j+1)2^{-n}} f(x)dx$$

Show that $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. with respect to the Lebsgue measure.

Q4. If $X_1, X_2, \ldots, X_n, \ldots$ are independent random variables that are almost surely positive (i.e. $P[X_i > 0] = 1$) with $E[X_i] = 1$, show that

$$Z_n = X_1 X_2 \cdots X_n$$

is a martingale. What can you say about

$$\lim_{n\to\infty} Z_n = Z?$$

When is Z nonzero? Is it sufficient if

$$\prod_i E[X_i^{-a}] < \infty$$

for some $a > 0$? Why?

Q5. Let $\{X_n\}$ be independent random variables where X_n is distributed according to a Gamma distribution with density $f_n(x)$ given by

$$f_n(x) = \frac{\alpha_n^{p_n}}{\Gamma(p_n)} e^{-\alpha_n x} x^{p_n - 1}$$

for ≥ 0 and 0 otherwise.

(a) Find necessary and sufficient conditions on α_n, p_n so that $\sum_n X_n$ converges almost surely.

(b) For $S_n = X_1 + X_2 + \cdots + X_n$ compute $E[S_n]$ and $\text{Var}[S_n]$.

(c) When does

$$\frac{S_n - E[S_n]}{\sqrt{Var[S_n]}}$$

have a limiting distribution that is the standard normal distribution?