

Probability Theory

S.R.S.Varadhan
Courant Institute of Mathematical Sciences
New York University

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Preface

These notes are based on a first year graduate course on Probability and Limit theorems given at Courant Institute of Mathematical Sciences. Originally written during 1997-98, they have been revised during academic year 1998-99 as well as in the Fall of 1999. I want to express my appreciation to those who pointed out to me several typos as well as suggestions for improvement. I want to mention in particular the detailed comments from Professor Charles Newman and Mr Enrique Loubet. Chuck used it while teaching the course in 98-99 and Enrique helped me as TA when I taught out of these notes again in the Fall of 99. These notes cover about three fourths of the course, essentially discrete time processes. Hopefully there will appear a companion volume some time in the near future that will cover continuous time processes. A small amount measure theory that is included. While it is not meant to be complete, it is my hope that it will be useful.

