Chapter 6

Non-gradient systems

6.1 Two color system

Let us look at the situation where there are two types of particles. Type 1 and type 2. The state space is $\Omega_N = \{0, 1, 2\}^{N^d}$. We define

$$
\begin{aligned}
\zeta_1(x) &= 1 \text{ if there is a type 1 particle at } x \text{ and 0 otherwise}\\
\zeta_2(x) &= 1 \text{ if there is a type 2 particle at } x \text{ and 0 otherwise}\\
\eta(x) &= \zeta_1(x) + \zeta_2(x) = 1 \text{ if there is a particle at } x \text{ and 0 otherwise}\\
\eta(x) &= 0 \text{ if there is no particle at } x\\
\zeta(x) &= \{\zeta_1(x), \zeta_2(x)\}\\
\zeta &= \{\zeta(x)\}
\end{aligned}
$$

We have three empirical measures

$$
\begin{aligned}
\lambda_1(s, du) &= \frac{1}{N^d} \sum_x \delta_{\frac{x}{N}} \zeta_1(s, x)\\
\lambda_2(s, du) &= \frac{1}{N^d} \sum_x \delta_{\frac{x}{N}} \zeta_2(s, x)\\
\lambda(s, du) &= \frac{1}{N^d} \sum_x \delta_{\frac{x}{N}} \eta(s, x)
\end{aligned}
$$

$$
\lambda(s, du) = \lambda_1(s, du) + \lambda_2(s, du)
$$

The evolution is specified by the generator of the Markov process quite similar to the old one of a single type.

$$(N^2\mathcal{A}_N F)(\zeta_1,\zeta_2) = N^2 \sum_{x,y\in\mathbb{Z}_N^d} \pi(y-x)\eta(x)(1-\eta(y))[F(\zeta^{x,y}) - F(\zeta)]$$
$$= \frac{N^2}{2} \sum_{x,y\in\mathbb{Z}_N^d} \pi(y-x)a_{x,y}(\zeta)[F(\zeta^{x,y}) - F(\zeta)]$$

where

$$a_{x,y}(\zeta) = [\eta(x)(1-\eta(y)) + \eta(y)(1-\eta(x))$$

$a_{x,y}(\zeta)$ is either 0 or 1, and when it is 1, one of the two sites is empty and a jump from x to y or y to x can occur with equal rate $\pi(y-x)$.

The particles evolve as before and are not affected by their type. But we keep track of their type. Let $k_1 = k_1(N), k_2 = k_2(N)$ and $k(N) = k_1(N) + k_2(N)$ be respectively the number of particles of type 1, type 2 and of either type. If $k(N) \le N^d - 1$, i.e if there is at least one empty site, then the only invariant distribution for the Markov process is the uniform distribution $\mu_{N,k_1(N),k_2(N)}$ over all possible configurations. In the limit, assuming $k_r(N)N^{-d} \to \rho_r$ for $r = 1,2$ one has a product measure μ_{ρ_1,ρ_2}, with each site having independently a particle of type 1 with probability ρ_1, type 2 with probability ρ_2 and being empty with probability $1-\rho$ where $\rho = \rho_1 + \rho_2$. The situation with $\rho = 1$ is the other extreme, where there is no movement and every configuration is static..

There are Dirichlet forms associated with these processes given by

$$\langle -\mathcal{A}_N f, f\rangle_{k_1,k_2} = \mathcal{D}^N_{k_1,k_2}(f) = \frac{1}{4}E^{\mu_{N,k_1,k_2}}\left[\sum_{x,y\in\mathbb{Z}_N^d} a_{x,y}(\zeta)\pi(y-x)[f(\zeta^{x,y}) - f(\zeta)]^2\right]$$

and the similar form

$$\mathcal{D}_{\rho_1,\rho_2}(f) = \frac{1}{4}E^{\mu_{\rho_1,\rho_2}}\left[\sum_{x,y\in\mathbb{Z}^d} a_{x,y}(\zeta)\pi(y-x)[f(\zeta^{x,y}) - f(\zeta)]^2\right]$$

on $\mathbb{Z}^d$.

We can also consider our process in a box $\mathbb{B}_q$ of size $(2q+1)^d$ without assuming a periodic boundary. Jumping outside the box is not allowed. In this case a minimal number n_0, that depends only on $\pi(\cdot)$, of empty sites are needed to ensure uniqueness of the uniform distribution μ_{q,k_1,k_2} as the only invariant distribution. If $\pi(\pm e_i) > 0$ for all i, then n_0 can be taken to be 1. The operator and the Dirichlet form look identical except x, y are now restricted to $\mathbb{B}_q$.

We take two smooth test functions $J^{(1)}(u)$ and $J^{(2)}(u)$ on the torus $\mathcal{T}^d$ and consider

$$F_{J^{(1)},J^{(2)}}(\zeta) = \frac{1}{N^d} \sum_{x \in \mathbb{Z}_N^d} [J^{(1)}(\frac{x}{N})\zeta_1(x) + J^{(2)}(\frac{x}{N})\zeta_2(x)]$$

Next we compute $(N^2 \mathcal{A}_N F_{J^{(1)},J^{(2)}})(\zeta)$ as

$$N^{2-d} \sum_{x,y} \pi(y-x)\big[\zeta_1(x)(1-\eta(y))[J^{(1)}(\frac{y}{N}) - J^{(1)}(\frac{x}{N})] + \zeta_2(x)(1-\eta(y))[J^{(2)}(\frac{y}{N}) - J^{(2)}(\frac{x}{N})]\big]$$

and can approximate it, using the symmetry of $\pi(z)$ by

$$\begin{aligned}
\frac{N^{1-d}}{2} \sum_{x,y} \pi(y-x)\big[&\zeta_1(x)(1-\eta(y))(\nabla J^{(1)}(\frac{y}{N}) + \nabla J^{(1)}(\frac{x}{N})) \cdot (y-x) \\
&+ \zeta_2(x)(1-\eta(y))(\nabla J^{(2)}(\frac{y}{N}) + \nabla J^{(2)}(\frac{x}{N})) \cdot (y-x)] \\
= \frac{N}{N^d} \sum_x &[(\nabla J^{(1)})(\frac{x}{N}) \cdot \mathbf{f}^1(\tau_x \zeta) + (\nabla J^{(2)})(\frac{x}{N}) \cdot \mathbf{f}^2(\tau_x \zeta)]
\end{aligned}$$

where for $r = 1, 2$

$$\mathbf{f}_i^r(\zeta) = \frac{1}{2} \sum_z \pi(z) < z, e_i > [\zeta_r(0)(1-\eta(z)) - \zeta_r(z)(1-\eta(0))]$$

The factor N is a problem which will not go away. Can not do a summation by parts to bring in the second difference. $\mathbf{f}_i^1$ and $\mathbf{f}_i^2$ are not gradients. Note that their sum

$$\begin{aligned}
\mathbf{f}_i^1 + \mathbf{f}_i^2 &= \frac{1}{2} \sum_z \pi(z) < z, e_i > [\eta(0)(1-\eta(z)) - \eta(z)(1-\eta(0))] \\
&= \frac{1}{2} \sum_z \pi(z) < z, e_i > [\eta(0) - \eta(z)] \\
&= \frac{1}{2} \sum_z \pi(z) < z, e_i > [\eta(0) - (\tau_z \eta)(0)]
\end{aligned}$$

is a "gradient" that allows another summation by parts to get rid of the unwelcome factor of an extra N when there is only one type of particle.

The expectations of $\mathbf{f}_i^1$ and $\mathbf{f}_i^2$ in any equilibrium μ_{ρ_1,ρ_2} are easily calculated and equal 0. We need to understand this combination of large N and currents $\mathbf{f}$ that are small on the average. We will determine constants $\{c_{i,j}^{r,r'}\}$ with $r, r' = 1, 2$ and $1 \le i, j \le d$, (that are actually functions of ρ_1, ρ_2) such that

$$\mathbf{f}_i^r + \frac{1}{2} \sum_{r',j} c_{i,j}^{r,r'} [\zeta_{r'}(e_j) - \zeta_{r'}(0)] = w_i^r$$

and $\{w_i^r\}$ are negligible. The sense in which they are negligible has to be specified. Unlike in the gradient case they will become negligible only when integrated over time. The context will be relative to process in equilibrium under the measure μ_{ρ_1,ρ_2}. This makes the constants $c_{i,j}^{r,r'}$ functions of ρ_1, ρ_2. We can now do another summation by parts, get rid of the extra N, replacing $\{N(\zeta_r'(e_j) - \zeta_r'(0))\}$ by $\frac{\partial \rho_{r'}}{\partial u_j}$. We end up with a weak formulation

$$\frac{\partial}{\partial t}\sum_r \langle J_r, \rho_r\rangle - \frac{1}{2}\sum_r \langle J_r, \sum_{r'i,j} \frac{\partial}{\partial u_i} c_{i,j}^{r,r'}(\rho_1,\rho_2)\frac{\partial \rho_{r'}}{\partial u_j}\rangle = 0$$

of the elliptic system

$$\frac{\partial \rho_r}{\partial t} = \frac{1}{2}\sum_{r',i,j} \frac{\partial}{\partial u_i} c_{i,j}^{r,r'}(\rho_1,\rho_2)\frac{\partial \rho_j}{\partial u_j}$$

6.2 Approximations.

There are three versions of our basic simple exclusion process with two colors. On all of $\mathbb{Z}^d$, on $\mathbb{Z}_N^d$, with periodic, or reflecting boundary conditions. Their generators are

$$(\mathcal{A}f)(\zeta) = \sum_{x,y\in\mathbb{Z}^d} \eta(x)(1-\eta(y))\pi(y-x)[f(\zeta^{x,y}) - f(\zeta)]$$

$$(\mathcal{A}_N f)(\zeta) = \sum_{x,y\in\mathbb{Z}_N^d} \eta(x)(1-\eta(y))\pi(y-x)[f(\zeta^{x,y}) - f(\zeta)]$$

In a finite box $\mathbb{B}_q = [-q,q]^d$ of size $(2q+1)^d$, with reflecting boundary conditions, the generator will be

$$(\mathcal{A}_q^o f)(\zeta) = \sum_{|x|,|y|\le q} \eta(x)(1-\eta(y))\pi(y-x)[f(\zeta^{x,y}) - f(\zeta)]$$

We have the three Dirichlet forms.

$$\mathcal{D}(u) = \frac{1}{4}E^{\mu_{\rho_1,\rho_2}}\Big[\sum_{x,y\in\mathbb{Z}^d} a_{x,y}(\zeta)\pi(x,y)[u(\zeta^{x,y}) - u(\zeta)]^2\Big]$$

$$\mathcal{D}_N(u) = \frac{1}{4}E^{\mu_{N,k_1,k_2}}\Big[\sum_{x,y\in\mathbb{Z}_N^d} a_{x,y}(\zeta)\pi(x,y)[u(\zeta^{x,y}) - u(\zeta)]^2\Big]$$

$$\mathcal{D}_q^o(u) = \frac{1}{4}E^{\mu_{q,k_1,k_2}}\Big[\sum_{|x|,|y|\le q} a_{x,y}(\zeta)\pi(x,y)[u(\zeta^{x,y}) - u(\zeta)]^2\Big]$$

We have three types of local functions all having the common property that they have mean 0 under every μ_{ρ_1,ρ_2}. The first type consists of functions $f = \mathcal{A}u$ for some local function u. As u varies, we get a large family $\mathcal{N}$ of local functions $\{f\}$. The second family of "currents" consists of $2d$ functions $\{\mathbf{f}_i^a\}$, given by

$$\begin{aligned}\mathbf{f}_i^r(\zeta) &= \frac{1}{2}\sum_z \pi(z) < z, e_i > [\zeta_r(0)(1-\eta(z)) - \zeta_r(z)(1-\eta(0))] \\ &= \frac{1}{2}\sum_z \pi(z) < z, e_i > a_{0,z}(\zeta)[\zeta_r(0) - \zeta_r(z)]\end{aligned}$$

where $a_{x,y}(\zeta) = \eta(x)(1-\eta(y)) + \eta(y)(1-\eta(x))$. Both families have the property that their expectation is 0, under every invariant distribution in every sufficiently large box. If u is defined in a box then $\mathcal{A}u$ has zero mean with respect to any invariant distribution in any box $\mathbb{B}_q$ provided $\mathcal{A}_q u = \mathcal{A}u$. Finally we have the $2d$ microscopic "density gradients" $\mathbf{d}_i^r = \{\zeta_r(e_i) - \zeta_b(0)\}$. The first type will be "negligible". The goal is to express the "currents" as a linear combination of the third type, "density gradients" modulo the first type that are "negligible". The density gradients are a bit more difficult to handle because, their expectation is not 0 if the density is 1. There will be problems when ρ is close to 1. But the basic object we want to approximate is 0 if the density is 1, so there is a natural cutoff when ρ is c;use to 1.

We consider a function of the form $f = \mathcal{A}u = \mathcal{A}_N u$ where u is a local function. Let $U(\zeta) = \sum_{x\in\mathbb{Z}_N^d} u(\tau_x\zeta)$ and $F(\zeta) = \sum_{x\in\mathbb{Z}_N^d} f(\tau_x\zeta)$. In the speeded up time scale with generator $N^2\mathcal{A}$,

$$\frac{N^2}{N^d}\int_0^t F(\zeta(s))ds = \frac{1}{N^d}[U(\zeta(t)) - U(\zeta(0))] - M_N(t)$$

The quadratic variation of the martingale term $M_N(t)$ is of the order $N^d \times N^2 \times N^{-2d} = N^{2-d}$. More over $\frac{1}{N^d}[U(\zeta(t)) - U(\zeta(0))]$ is uniformly bounded. Therefore

$$\frac{1}{N^d}\int_0^t [NF](\zeta(s))ds = q_N^1(t) + q_N^2(t)$$

where $|q_N^1(t)| \le \frac{C}{N}$ and q_N^2 is a Martingale with jumps of size $N^{-(d+1)}$ with quadratic variation tN^{-d}. This makes them "negligible".

6.3 How do we proceed and what do we need?

For any smooth function A we need to be able to replace

$$\frac{1}{N^d}\int_0^t [\sum_{x\in\mathbb{Z}_N^d} A(\frac{x}{N})(N\mathbf{f}_i^r)(\tau_x\zeta(s))]ds$$

by

$$-\frac{1}{2N^d}\int_0^t[\sum_{x\in\mathbb{Z}_N^d}A(\frac{x}{N})\sum_{r',j}c_{i,j}^{r,r'}(\bar{\zeta}_{1,x,N\epsilon'}(s),\bar{\zeta}_{2,x,N\epsilon'}(s))\frac{1}{2\epsilon}[\bar{\zeta}_{r',x+N\epsilon e_j,N\epsilon'}(s)-\bar{\zeta}_{r',x-N\epsilon e_j,N\epsilon'}(s)]ds$$

with an error that becomes negligible as $N\to\infty$ followed by $\epsilon,\epsilon'\to 0$. That would lead to

$$-\frac{1}{2}\int_0^t\int_{\mathcal{T}^d}A(u)\sum_{r',j}c_{i,j}^{r,r'}(\bar{\rho}_{1,\epsilon'}(s,u),\bar{\rho}_{1,\epsilon'}(s,u))[\frac{1}{2\epsilon}[\bar{\rho}_{r,\epsilon'}(s,u+\epsilon e_j)-\bar{\rho}_{r,\epsilon'}(s,u-\epsilon e_j)]ds$$

where

$$\bar{\rho}_{r,\epsilon'}(s,u)=\frac{1}{(2\epsilon')^d}\int_{|v-u|\le\epsilon'}\rho_r(s,v)dv$$

and

$$\bar{\zeta}_{r,x,N\epsilon'}=\frac{1}{(2N\epsilon')^d}\sum_{|y-x|\le N\epsilon'}\zeta_r(x)$$

As $\epsilon',\epsilon\to 0$ this becomes

$$-\frac{1}{2}\int_0^t\int_{\mathcal{T}^d}A(u)\sum_{r',j}c_{i,j}^{r,r'}(\rho_1(s,u),\rho_2(s,u))\frac{\partial\rho_{r'}(s,u)}{\partial u_j}ds$$

If we consider a family of local function $v(\rho_1,\rho_2,\zeta)$ and $f(\rho_1,\rho_2,\cdot)=\mathcal{A}v(\rho_1,\rho_2,\cdot)$ depending smoothly on ρ_1,ρ_2

$$\begin{aligned}\frac{1}{N^d}\int_0^t&\sum_{x\in\mathbb{Z}_N^d}(Nf)(\tau_x\bar{\zeta}_{1,N\epsilon'}(t),\bar{\zeta}_{1,N\epsilon'},\tau_x\zeta(t))\\&=\frac{1}{N^{d+1}}\sum_{x\in\mathbb{Z}_N^d}[v(\tau_x\bar{\zeta}_{1,N\epsilon'}(t),\bar{\zeta}_{1,N\epsilon'},\tau_x\zeta(t))-v(\tau_x\bar{\zeta}_{1,N\epsilon'}(0),\bar{\zeta}_{1,N\epsilon'},\tau_x\zeta(0))]+M_N(t)+o(1)\end{aligned}$$

is negligible. We need to show

$$\inf_{f(\cdot,\cdot,\cdot)}\limsup_{\epsilon,\epsilon'\to}\limsup_{N\to\infty}\frac{1}{N^d}\log E^{N,k_1(N),k_2(N)}[\exp[N\int_0^t\Theta_N(f,\zeta(s))ds]]=0$$

where

$$\begin{aligned}\Theta_N(f,\zeta)&=\sum_{x\in\mathbb{Z}_N^d}A(\frac{x}{N})[\mathbf{f}_i^r(\tau_x\zeta)-f(\tau_x\bar{\zeta}_{1,N\epsilon'},\tau_x\bar{\zeta}_{2,N\epsilon'},\tau_x\zeta)+T(\tau_x\bar{\zeta}_{1,N\epsilon'},\tau_x\bar{\zeta}_{2,N\epsilon'}\tau_x\zeta)]\\&=\sum_{x\in\mathbb{Z}_N^d}A(\frac{x}{N})\theta(\tau_x,\zeta)\end{aligned}$$

and

$$T(\bar{\zeta}_{1,N\epsilon'},\bar{\zeta}_{2,N\epsilon'}\zeta)=\frac{1}{2}\sum_{r',j}c_{i,j}^{r,r'}(\tau_x\bar{\zeta}_{1,N\epsilon'},\tau_x\bar{\zeta}_{2,N\epsilon'})\frac{1}{2\epsilon}[\tau_{x+N\epsilon e_j}\bar{\zeta}_{r',N\epsilon'}-\tau_{x-N\epsilon e_j}\bar{\zeta}_{r',N\epsilon'}]$$

By the use of the variational formula and Feynman-Kac representation

$$\begin{aligned}
&\frac{1}{N^d}\log E^{N,k_1(N),k_2(N)}[\exp[N\int_0^t\Theta_N(f,\zeta(s))ds]]\\
&\le tN^{-d}\sup_G\left[NE^{\mu_{N,k_1,k_2}}\left[\Theta_N(f,\zeta)G^2\right]-N^2\mathcal{D}_N(G)\right]\\
&=tN^{2-d}\sup_G\left[N^{-1}E^{\mu_{N,k_1,k_2}}\left[\Theta_N(f,\zeta)G^2\right]-\mathcal{D}_N(G)\right]\\
&\simeq tN^{-d}\langle\Theta_N(f,\zeta),\Theta_N(f,\zeta)\rangle_{CLT}\\
&=tN^{-d}\sup_G\left[E^{\mu_{N,k_1,k_2}}\left[\Theta_N(f,\zeta)G\right]-\frac{1}{8}\mathcal{D}_N(G)\right]
\end{aligned}$$

If we have an expression of the form

$$E^{\mu_{N,k_1,k_2}}\left[G\sum_{x\in\mathbb{Z}_N^d}H(\tau_x\zeta)\right]-D_N(G)$$

to estimate, and if $H(\tau_x\zeta)$ allows us to do an integration by parts, i.e is made up of antisymmetric pieces we can rewrite the above expression as

$$E^{\mu_{N,k_1,k_2}}\left[\sum_{x,y\in\mathbb{Z}_N^d}[H_{x,y}(\zeta^{x,y})-H_{x,y}(\zeta)][G(\zeta_{x,y})-G(\zeta)]\right]-D_N(G)$$

Both sides can be localized by breaking them up into sums over $\mathbb{B}_k$. One can replace densities over small macroscopic blocks by densities over large microscopic blocks. The problems with $\rho_1+\rho_2\simeq 1$ has to be handled. The Dirichlet form $\mathcal{D}_N$ can be thought of as $(2k+1)^{-d}\sum_{x\in\mathbb{Z}_N^d}\mathcal{D}^0_{x,k}$. The problem can now be reduced to estimating the quantity (after trimming the edges!)

$$\Lambda(k,f)=\sup_G\left[E^{\mu_{k,k_1,k_2}}\left[G\sum_{x\in\mathbb{B}_k}[\mathbf{f}_i^r(\tau_x\zeta)-\sum_{j,r'}c_{i,j}^{r,r'}[\zeta_r(x+e_i)-\zeta_r(e_i)]-f(\tau_x\zeta)]\right]-\mathcal{D}^0_{k,k_1,k_2}(G)\right]$$

with $f=\mathcal{A}u$ for a local function u, and showing that with the proper choice of constants $c_{i,j}^{r,r'}(\rho_1,\rho_2)$,

$$\inf_u\limsup_{\substack{k\to\infty\\(2k+1)^{-d}k_r\to\rho_r}}(2k+1)^{-d}\Lambda(k,f)=0$$

6.4 Calculating variances.

Given two local functions g_1, g_2 depending on configurations in a box $\mathbb{B}_q$ and having mean 0 under every invariant distribution μ_{k,k_1,k_2} in $\mathbb{B}_k$ under $\mathcal{A}^o_k$, we try to define an inner product $[g_1, g_2]_{\rho_1,\rho_2}$ in two steps.

$$< g_1, g_2 >_{k,k_1,k_2} = \lim_{t\to\infty} \frac{1}{t} E^{\mu_{k,k_1,k_2}} \left[\int_0^t \sum_{|x|\le k-q} g_1(\tau_x \zeta(s))ds \int_0^t \sum_{|x|\le k-q} g_2(\tau_x \zeta(s))ds \right]$$

$$[g_1, g_2]_{\rho_1,\rho_2} = \lim_{\substack{k\to\infty \\ k_r(2k+1)^{-d}\to\rho_r}} (2k+1)^{-d} < g_1, g_2 >_{k,k_1,k_2}$$

We will construct, for each ρ_1, ρ_2 with $\rho_1 + \rho_2 < 1$, a Hilbert space $\mathcal{H} = \mathcal{H}_{\rho_1,\rho_2}$ and a map $g \to \hat{g} = \sigma(g)$ that imbeds linear combinations $g = \mathcal{A}u + \sum_{i,r} c^r_i \mathbf{f}^A_i + \sum_{i,r} \bar{c}^r_i \mathbf{d}^r_i$ with inner product $[g_1, g_2]_{\rho_1,\rho_2}$ isometrically into $\mathcal{H}_{\rho_1,\rho_2}$ with inner product $\langle\langle , \rangle\rangle$. It will turn out that $\sigma(\mathcal{A}u) \perp \sigma(\mathbf{d}^i_r)$, and $\mathcal{H}$ is spanned by them. Then the approximation is basically a projection.

$\mathcal{H}$ will be a subspace of maps $\hat{g} : \mathbb{Z}^d - \{0\} \to L_2[\mu_{\rho_1,\rho_2}]$ with inner product

$$\langle\langle \hat{g}_1, \hat{g}_2 \rangle\rangle = \frac{1}{2} E^{\mu_{\rho_1,\rho_2}} [\sum_z a_{0,z}(\zeta)\pi(z)\hat{g}_1(z,\zeta)\hat{g}_2(z,\zeta)]$$

If $f = \mathcal{A}u = \mathcal{A}^0_q u$ for some local u depending on $\mathbb{B}_q$, since $\mathcal{A}$ is linear and translation invariant,

$$\mathcal{A}^0_k \big(\sum_{|x|\le k-q} u(\tau_x \zeta) \big) = \sum_{|x|\le k-q} f(\tau_x \zeta)$$

and

$$\langle f, f \rangle_{k,k_1,k_2} = 2\mathcal{D}_N (\sum_{|x|\le k-q} u(\tau_x \zeta))$$

It is now easy to calculate the limit as $k \to \infty$, if $\mathcal{A}u_i = f_i$ and for $r = 1, 2$ $(2k+1)k_r \to \rho_r$

$$[f_1, f_2]_{\rho_1,\rho_2} = \frac{1}{2} E^{\mu_{\rho_1,\rho_2}} [\sum_z a_{0,z}(\zeta)\pi(z)\hat{f}^1(z,\zeta)\hat{f}^2(z,\zeta)] = \langle\langle \hat{f}_1, \hat{f}_2 \rangle\rangle$$

where for $i = 1, 2$

$$\hat{f}^i(z,\zeta) = \mathbf{U}^{\mathbf{i}}(\zeta^{0,z}) - \mathbf{U}^i(\zeta)$$

and

$$\mathbf{U}^i(\zeta) = \sum_{x\in\mathbb{Z}^d} u^i(\tau_x \zeta)$$

Although $\mathbf{U}^i$ are not well defined, $\hat{f}^i_{x,y} = \mathbf{U}^i(\zeta^{x,y}) - \mathbf{U}^i(\zeta)$ are well defined and satisfy linear identities. $\hat{f}^i_{x,y}(\zeta)$ are covariant, i.e. $\hat{f}^i_{x+z,y+z}(\zeta) = \hat{f}^i_{x,y}(\tau_z\zeta)$. If $\{\sigma_j\}$ are permutations of the form $x \leftrightarrow x + e_i$ for some i, and $\sigma_1\sigma_2\cdots\sigma_k = Id$, then with $\sigma_j = x \leftrightarrow x + e_{\alpha(j)}$ for some $e = e_{\alpha(j)}$

$$0 = \sum_{j=1}^{k} \mathbf{U}^i(\sigma_j\sigma_{j-1}\cdots\sigma_1\zeta) - \mathbf{U}^i(\sigma_{j-1}\cdots\sigma_1\zeta) = \sum_{i=1}^{k} \hat{f}^i_{x_j,x+e_{\alpha(j)}}(\sigma_{j-1}\cdots\sigma_1\zeta)$$

The Hilbert space $\mathcal{H}$ consists of all such maps $h_{0,z}(\zeta)$ such that $h_{x,y}(\zeta) = h_{0,y-x}(\tau_x\zeta)$ satisfy these linear identities, with inner product

$$\langle\langle h_1, h_2\rangle\rangle_{\rho_1,\rho_2} = \frac{1}{2}E^{\mu_{\rho_1,\rho_2}}\left[\sum_z a_{0,z}(\zeta)\pi(z)h^1(z,\zeta)h^2(z,\zeta)\right]$$

and $\mathcal{H}_0 \subset \mathcal{H}$ is the closure of the span of $\hat{f}$ as f ranges over $\mathcal{N}$. We need to consider two families of functions that are not in $\mathcal{N}$. $\{\mathbf{f}^r_i\}$ and $\zeta_r(e_i) - \zeta_r(0)$, with r equal to 1 or 2 and $1 \le i \le d$. We will show that they can be imbedded in $\mathcal{H}$ as well. Imbedding $\mathbf{f}^r_i$ is relatively easy. We can take

$$V^r_i(\zeta) = \sum_{x\in\mathbb{B}^d_k} < x, e_i > \zeta_r(x)$$

A calculation of $\mathcal{A}^o_k V^r_i$ yields

$$(\mathcal{A}^o_k V^r_i)(\zeta) = \sum_{x,y\in\mathbb{Z}_k} \zeta(x)(1-\eta(y))\pi(y-x) < y-x, e_i > \simeq \sum_x \mathbf{f}^r_i(\tau_x\zeta)$$

the error coming entirely from boundary terms. They can be controlled and become negligible for large k. Therefore

$$\sigma(\mathbf{f}^r_i)(z,\zeta) = V^r_i(\zeta^{0,z}) - V^r_i(\zeta) = < z, e_i > [\zeta_r(0) - \zeta_r(z)]$$

While we defer saying anything about $\sigma(\mathbf{d}^r_i)$ till later we can compute its inner product in $\mathcal{H}$ with objects in $\mathcal{H}_0$ and $\sigma(\mathbf{f}^r_i)$.

$$< f, \zeta_r(e_i) - \zeta_r(0) >_{k,k_1,k_2} = -2E^{\mu_{k,k_1,k_2}}\left[[\sum_{|x|\le k-q} u(\tau_x\zeta)][\sum_{|x|\le k-1}(\zeta_r(x+e_i) - \zeta_r(x))]\right] = O(k^{d-1})$$

The summation $\sum_{|x|\le k-1}(\zeta_r(x+e_i) - \zeta_r(x))$ telescopes, μ_{k,k_1,k_2} is almost a product measure and u is local. Therefore only the boundary contributes. This proves that $\sigma(\mathbf{f}^r_i) \perp \sigma(f)$ for all $f = \mathcal{A}u$, i.e. $\sigma(\mathbf{f}^r_i) \perp \mathcal{H}_0$.

We next compute the inner product, $\langle\langle\sigma(\mathbf{f}^r_i), \sigma(\mathbf{d}^{r'}_j)\rangle\rangle_{\rho_1,\rho_2}$. We can compute it as

$$
\begin{aligned}
&\langle\langle \sigma(\mathbf{f}_i^r), \sigma(\mathbf{d}_j^{r'})\rangle\rangle_{\rho_1,\rho_2} \\
&\quad = -2 \lim_{\substack{k\to\infty \\ (2k+1)^{-d}k_r \to \rho_r}} \frac{1}{(2k+1)^d} E^{\mu_{k,k_1,k_2}}\,[[\sum_x < x, e_i > \zeta_r(x)][\sum_{x_i = k, |x|\le k} \zeta_{r'}(x) - \sum_{x_i=-k,|x|\le k} \zeta_{r'}(x)]] \\
&\quad = -2[\delta_{r,r'}\rho_r - \rho_r\rho_{r'}]
\end{aligned}
$$

We next do a calculation. With $f_{0,z} = U(\zeta^{0,z}) - U(\zeta)$ and $U(\zeta) = \sum_{x\in\mathbb{Z}_N^d} u(\tau_x\zeta)$,

$$
\begin{aligned}
\inf_u \frac{1}{2} E^{\mu_{\rho_1,\rho_2}}\,[\sum_z a_{0,z}(\zeta)\pi(z)[< z, w > (a_1(\zeta_1(z) - \zeta_1(0)) + a_2(\zeta_2(z) - \zeta_2(0)) - f_{0,z}(\zeta)]^2] \\
= \langle w, S(\rho)w\rangle\langle a, R_1 a\rangle + \langle w, Dw\rangle\langle a, R_2 a\rangle
\end{aligned}
$$

where

$$
R_1 = \begin{pmatrix} \frac{\rho_1^2}{\rho} & -\frac{\rho_1\rho_2}{\rho} \\ -\frac{\rho_1\rho_2}{\rho} & \frac{\rho_2^2}{\rho}) \end{pmatrix}
$$

and

$$
R_2 = \begin{pmatrix} \rho_1^2\frac{1-\rho}{\rho} & \rho_1\rho_2\frac{1-\rho}{\rho} \\ \rho_1\rho_2\frac{1-\rho}{\rho} & \rho_2^2\frac{1-\rho}{\rho}) \end{pmatrix}
$$

Because the semigroup leaves the class expressions linear in ζ_1, ζ_2 with coefficients that are functions of η invariant, we can restrict $u(\zeta)$ to functions of the form $\sum_x \zeta_1(x)\psi_1(\tau_x\eta) + \zeta_2(x)\psi_2(\tau_x\eta)$. It better to choose instead of ζ_1, ζ_2 the combinations $\eta = \zeta_1 + \zeta_2$ and $\chi = \frac{\rho_2}{\rho}\zeta_1 - \frac{\rho_1}{\rho}\zeta_2$ that are orthogonal. ψ_1, ψ_2 do not depend on $\eta(0)$. In terms of χ and η, using the orthogonality the variational problem reduces to

$$
\langle w, Dw\rangle\langle a, R_2 a\rangle + \inf_\psi \frac{1}{2} E^{\mu_{\rho_1,\rho_2}}[\sum_z p(z)a_{0,z}[\langle z, w\rangle(\chi(z) - \chi(0)) - D^{0,z}(\sum_x \chi(x)\psi(\tau_x\eta)]^2]
$$

The second term is simplified to give

$$
\begin{aligned}
&\frac{\rho_1\rho_2}{\rho}(a_1 - a_2)^2 \inf_\psi E^{\mu_\rho}[\sum_z p(z)(1-\eta(z))[\langle z, w\rangle - \psi(T_z\eta) + \psi(\eta)]^2 + \sum_{x,y} \pi(y-x)[\psi(\eta^{xy}) - \psi(\eta)]^2] \\
&\quad = \langle w, S(\rho)w\rangle\langle a, R_1 a\rangle
\end{aligned}
$$

The variational formula for $S(\rho)$ is easily checked by doing the variation explicitly.

The last calculation is to show that $\sigma(\mathbf{f}_i^r) + \sum_{j,r'} c_{j,r'}^{i,r}\sigma(\mathbf{d}_j^{r'}) \in \mathcal{H}_0$ for a suitable choice of $c_{j,r'}^{i,r}$ and determine them. If we denote the projections of $\sigma(\mathbf{f}_i^r)$ in the orthogonal complement of $\mathcal{H}_0$ by $\widehat{f}_i^r$ then we know that

$$
A_{j,r'}^{i,r}(\rho) = \frac{1}{2}\langle\langle \widehat{f}_i^r, \widehat{f}_j^{r'}\rangle\rangle = S_{i,j}(\rho)(\rho_r\delta_{r,r'} - \frac{\rho_r\rho_{r'}}{\rho}) + D_{i,j}\rho_r\rho_{r'}\frac{1-\rho}{\rho}
$$

we also know

$$\frac{1}{2}\langle\langle \widehat{f}_i^r, \mathbf{d}_j^{r'} \rangle\rangle = \frac{1}{2}\langle\langle \sigma(\mathbf{f}_i^r), \mathbf{d}_j^{r'} \rangle\rangle = -[\delta_{r,r'}\rho_r - \rho_r\rho_{r'}] = -\chi^{-1}_{r,r'}$$

By elementary calculation

$$C = A\chi$$

Where

$$A = S \otimes R_1 + I \otimes R_2$$

and

$$\chi = \begin{pmatrix} \frac{1}{\rho_1} + \frac{1}{1-\rho} & \frac{1}{1-\rho} \\ \frac{1}{1-\rho} & \frac{1}{\rho_2} + \frac{1}{1-\rho} \end{pmatrix}$$