Chapter 4

Hydrodynamic Scaling

4.1 From Classical Mechanics to Euler Equations.

The basic example of hydrodynamical scaling is naturally hydrodynamics itself. Let us start with a collection of $N \simeq \bar{\rho}\ell^3$ classical particles in a large periodic cube Λ_ℓ of side ℓ in R^3. The motion of the particles are governed by the equations of motion of a classical Hamiltonian dynamical system with energy given by

$$H(p\,,q) = \frac{1}{2}\sum_{i=1}^{N} \|p_i\|^2 + \frac{1}{2}\sum_{i\neq j} V(q_i - q_j) \tag{1.1}$$

Here, $q_i \in \Lambda_\ell$ is the position of the i-th particle and $p_i \in R^3$ is its velocity and $k = 1, 2, 3$ refer to the three components of position or velocity. The repulsive potential $V \geq 0$ is even and has compact support in R^3. The interaction in particular is short range. The equations of motion are

$$\begin{cases} \frac{dq_i^k}{dt} = \frac{\partial H(p\,,q)}{\partial p_i^k} = p_i^k \\ \frac{dp_i^k}{dt} = -\frac{\partial H(p\,,q)}{\partial q_i^k} = -\sum_{i=1}^N V_k(q_i - q_j) \end{cases}$$

where

$$V_k(q) = \frac{\partial V(q)}{\partial q^k}$$

is the gradient of V. The dynamical system has five conserved quantities. The total number N of particles, the total momenta $\sum_{i=1}^N p_i^k$ for $k = 1, 2, 3$ and the total energy $H(p\,,q)$ given by (1.1). The hydrodynamic scaling in this context consists of rescaling space and time by a factor of ℓ. The rescaled space is the unit torus $\mathbf{T}^3$ in 3-dimensions. The macroscopic quantities to be studied correspond to conserved quantities. The first one of these is the density, and is measured by a function $\rho(t,x)$ of t and x. For each $\ell < \infty$ it is approximated by $\rho_\ell(t,x)$, defined by

$$\int_{\mathbf{T}^d} J(x)\rho_\ell(t,x)dx = \frac{1}{\ell^3}\sum_{i=1}^{N} J(\frac{q_i(\ell t)}{\ell})$$

A straight forward differentiation using (1.2) yields

$$\begin{aligned}\frac{d}{dt}\int_{\mathbf{T}^d} J(x)\rho_\ell(t,x)dx &= \frac{d}{dt}\frac{1}{\ell^3}\sum_{i=1}^{N} J(\frac{q_i(\ell t)}{\ell}) \\ &= \frac{1}{\ell^3}\sum_{i=1}^{N}(\nabla J)(\frac{q_i(\ell t)}{\ell})\cdot p_i(\ell t) \\ &\simeq \int_{\mathbf{T}^d}(\nabla J)(x)\cdot\rho_\ell(t,x)u(t,x)dx\end{aligned}$$

where $u = u^k$, $k = 1,2,3$ are the average velocity of the fluid. This introduces three other macroscopic variables, which represent three coordinates of the momenta that are conserved.

We can now write down the first of our five equations

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho u) = 0 \tag{1.3}$$

To derive the next three equations we start with test functions J and differentiate, again using (1.2), for $k = 1,2,3$

$$\frac{d}{dt}\frac{1}{\ell^3}\sum_{i=1}^{N} J(\frac{q_i(\ell t)}{\ell})p_i^k(\ell t) = \frac{1}{\ell^3}\sum_{i=1}^{N} p_i^k(\ell t)(\nabla J)(\frac{q_i(\ell t)}{\ell})\cdot p_i(\ell t) \tag{4.1}$$

$$-\frac{1}{\ell^2}\sum_{i=1}^{N}\sum_{j=1}^{N} J(\frac{q_i(\ell t)}{\ell})V_k(q_i(\ell t) - q_j(\ell t)) \tag{4.2}$$

The next step is rather mysterious and requires considerable explanation. Quantities

$$\sum_{i,j=1}^{N}\psi(q_i(t) - q_j(t))$$

and

$$\sum_{i=1}^{N} p_i^k\, p_i^r$$

are not conserved. They depend on spacings between particles or velocities of individual particles that change in the microscopic time scale and hence do so rapidly in the macroscopic time scale. They can therefore be replaced by their space-time averages. By appealing to an 'Ergodic Theorem' they can be replaced by ensemble averages with repect to their equilibrium distributions. The 'ensemble' consists of an infinite collection of points $\{p_\alpha, q_\alpha\}$, in the phase space $R^3 \times R^3$. There is a natural five parameter family of measures $\mu_{\rho,u,T}$ that are invariant under spatial translation as well as Hamiltonian dynamics. The points $\{p_\alpha\}$ are distributed according to a Gibbs Distribution with density ρ and formal interaction

$$\frac{1}{2T}\sum_{\alpha,\beta} V(q_\alpha - q_\beta)$$

In other words $\{q_\alpha\}$ is a point process obtained by taking infinite volume limit of $N = \ell^3\rho$ particles distributed in cube of side ℓ according to the joint density

$$\frac{1}{Z}\exp\big[-\frac{1}{2T}\sum_{i,j=1}^{N} V(q_i - q_j)\big]$$

where Z is the normalization constant. The velocities $\{p_\alpha\}$ are distributed independently of each other as well as of $\{q_\alpha\}$, having a common 3-dimensional Gaussian distribution with mean u and covariance TI. Assuming that the infinite volume limit exists in a reasonable sense it will be a point process defined as an infinite volume Gibbs measure $\mu_{\rho,T}$. The velocities $\{p_\alpha\}$ will be an independent Gaussian ensemble $\nu_{u,T}$.

The first term in (1.4) involves sums of $p_i^k p_i^r$ that are replaced by their expectations in the Gaussian ensemble $u^k u^r + T\delta_{k,r}$.

If we now use the skew-symmetry of $V_k = \frac{\partial V}{\partial q_k}$, we can rewrite the second term of (1.4) as

$$\begin{aligned}
&-\frac{1}{2\ell^2}\sum_{i=1}^{N}\sum_{j=1}^{N}(J(\frac{q_i(\ell t)}{\ell}) - J(\frac{q_j(\ell t)}{\ell})V_k(q_i(\ell t) - q_j(\ell t))\\
&\simeq -\frac{1}{2\ell^3}\sum_{i=1}^{N}\sum_{j=1}^{N} J_r(\frac{q_i(\ell t)}{\ell})(q_i^r(\ell t) - q_j^r(\ell t))V_k(q_i(\ell t) - q_j(\ell t))\\
&= \frac{1}{\ell^3}\sum_{i=1}^{N}\sum_{j=1}^{N} J_r(\frac{q_i(\ell t)}{\ell})\psi_k^r(q_i(\ell t) - q_j(\ell t))\\
&\simeq \int_{\mathbf{T}^d}\frac{\partial J}{\partial x_r}(x)\mathbf{P}_k^r(\rho(t,x), T(t,x))dx
\end{aligned}$$

Where $\mathbf{P}_k^r(\rho\,, T)$ is the 'pressure' per unit volume in the Gibbs ensemble

$$\mathbf{P}_k^r(\rho\,,T) = \lim_{\ell\to\infty} E^{\mu_{\rho,T}}\left\{\frac{1}{\ell^3}\sum_{\substack{|q_\alpha|\le\ell\\ |q_\beta|\le\ell}}\psi_k^r(q_\alpha - q_\beta)\right\}$$

We now integrate by parts, remove the test function J and obtain

$$\frac{d}{dt}(\rho u) + \nabla \cdot (\rho\, u \otimes u + \rho\, T I + \mathbf{P}(\rho\,, T)) = 0 \tag{1.5}$$

There is an equation of state that expresses the total energy per unit volume e as

$$e(\rho\,, u\,, T) = \frac{1}{2}\rho(|u|^2 + 3T) + f(\rho\,, T) \tag{1.6}$$

where $f(\rho\,, T)$, the potential energy per unit volume, is given by

$$f(\rho\,, T) = \lim_{\ell\to\infty} E^{\mu_{\rho,T}}\left\{\frac{1}{2\ell^3}\sum_{\substack{|q_\alpha|\le\ell\\|q_\beta|\le\ell}} V(q_\alpha - q_\beta)\right\}$$

Although we will not derive it, there is a similar equation for $e(t, x)$ that is obtained by differentiating

$$\frac{d}{dt}\frac{1}{2\ell^3}\sum_{i=1}^{N} J(\frac{q_i(\ell t)}{\ell})\Big[|p_i(\ell t)|^2 + \sum_{j=1}^{N} V(q_i(\ell t) - q_j(\ell t))\Big]$$

and proceeding in a similar fashion. It looks like

$$\frac{de}{dt} + \nabla \cdot \big[(e + T)u + \mathbf{P}(\rho\,, T)u\big] = 0 \tag{1.7}$$

The five equations ((1.3),(1.5) and (1.7)) in five variables (actually in six variables with one relation (1.6)) is a symmetrizable first order system of non-linear hyperbolic conservation laws. Given smooth initial data they have local solutions.

Rigorous derivation of these equations does not exist. The ergodic theory is definitely plausible, but hard to establish. The reason is that we have essentially an infinite system of ordinary differential equations representing a classical deterministic dynamical system and the ergodicity properties are nearly impossible to establish in any general context. However if we had some noise in the system, i.e. stochastic dynamics instead of deterministic dynamics, then we will be concerned with the ergodic theory of Markov Processes of some kind, which is far more accessible. This will be the focus of our future lectures.

4.2 Simple Exclusion Processes

The simplest example is a system of noninteracting particles undergoing independent motions. For instance we could have on $\mathcal{T}^3$, $k_N \simeq \bar{\rho}N^3$ particles, behaving collectively like independent Brownian Particles. If the initial configuration of the k_N particles is such that the empirical distribution

$$\nu_0(dx) = \frac{1}{N^3}\sum_i \delta_{x_i}$$

has a deterministic limit $\rho_0(x)dx$, then the empirical distribution

$$\nu_t(dx) = \frac{1}{N^3}\sum_i \delta_{x_i(t)}$$

of the configuration at time t, has a deterministic limit $\rho(t\,,x)dx$ as $N \to \infty$ and $\rho(t\,,x)$ can be obtained from $\rho_0(x)$ by solving the heat equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\Delta\rho$$

with the initial condition $\rho(0\,,x) = \rho_0(x)$. The proof is an elementary law of large numbers argument involving a calculation of two moments. Let $f(x)$ be a continuous function on $\mathbf{T}$ and let us calculate for

$$U = \frac{1}{N^3}\sum_i f(x_i(t))$$

the first two moments given the initial configuration $(x_1, \cdots, x_{k_N})$

$$E(U) = \frac{1}{N^3}\sum \int_{\mathbf{T}^3} f(y)p(t\,,x_i\,,y)dy$$

and an elementary calculation reveals that the expectation converges to the following constant.

$$\int_{\mathbf{T}^3}\int_{\mathbf{T}^3} f(y)p(t\,,x\,,y)\rho_0(x)dydx = \int_{\mathbf{T}^3} f(y)\rho(t\,,y)dy$$

The independence clearly provides a uniform upper bound of order N^{-3} for the variance that clearly goes to 0. Of course on $\mathbf{T}^3$ we could have had a process obtained by rescaling a random walk on a large torus of size N. Then the hydrodynamic scaling limit would be a consequence of central limit theorem for the scaling limit of a single particle and the law of large numbers resulting from the averaging over a large number of independently moving particles. The situation could be different if the particles interacted with each other.

The next class of examples are called simple exclusion processes. They make sense on any finite or countable set X and for us X will be either the integer lattice $\mathbf{Z}^d$ in d-dimensions or $\mathbf{Z}_N^d$ obtained from it as a quotient by considering each coordinate modulo N. At any given time a subset of these lattice sites will be occupied by partcles, with atmost one particle at each site. In other words some sites are empty while others are occupied with one particle. The particles move randomly. Each particle waits for an exponential random time and then tries to jump from the current site x to a new site y. The new site y is picked randomly according to a probability distribution $p(x\,,y)$. In particular $\sum_y p(x\,,y) = 1$ for every x. Of course a jump to y is not always possible. If the site is empty the jump is possible and is carried out. If the site already has a particle the jump cannot be carried out and the particle forgets about it and waits for another chance, i.e. waits for a new exponential waiting time.

If we normalize so that all waiting times have mean 1, the generater of the process can be written down as

$$(\mathcal{A} f)(\eta) = \sum_{x,y} \eta(x)(1-\eta(y))p(x\,,y)[f(\eta^{x,y}\,) - f(\eta)]$$

where η represents the configuration with $\eta(x) = 1$ if there is a particle at x and $\eta(x) = 0$ otherwise. For each configuration η and a pair of sites x, y the new configuration $\eta^{x,y}$ is defined by

$$\eta^{x,y}(z) = \begin{cases} \eta(y) & if \quad z = x \\ \eta(x) & if \quad z = y \\ \eta(z) & if \quad z \neq x\,, y \end{cases}$$

We will be concerned mainly with the situation where the set X is $\mathbf{Z}^d$ or $\mathbf{Z}^d_N$, viewed naturally as an Abelian group with $p(x\,,y)$ being translation invariant and given by $p(x\,,y) = \pi(y-x)$ for some probability distribution $\pi(\cdot)$. It is convenient to assume that π has finite support. There are various possibilities. $\pi(\cdot)$ is symmetric, i.e. $\pi(z) = \pi(-z)$ or more generally $\pi(\cdot)$ has mean zero, i.e. $\sum_z z\pi(z) = 0$, and finally $\sum_z z\,\pi(z) = m \neq 0$.

We shall first concentrate on the symmetric case. Let us look at the function

$$V_J(\eta) = \sum J(x)\eta(x)$$

and compute

$$\begin{aligned}(\mathcal{A}V_J)(\eta) &= \sum_{x,y} \eta(x)(1-\eta(y))\pi(y-x)(J(y)-J(x)) \\ &= \sum_{x,y} \eta(x)\pi(y-x)(J(y)-J(x)) \\ &= \sum_{x,y} \eta(x)[(\mathbf{P}-I)J](x) \\ &= V_{(\mathbf{P}-I)J}(\eta)\end{aligned}$$

The space of linear functionals is left invariant by the generator. It is not difficult to see that

$$E_\eta\big[V_J(\eta(t))\big] = V_{J(t)}(\eta)$$

where

$$J(t) = \exp[t(\mathbf{P}-I)]J$$

is the solution of

$$\frac{d}{dt}J(t\,,x) = (\mathbf{P}-I)J(t\,,x)$$

It is almost as if the interaction has no effect and in fact in the calculation of expectations of 'one particle' functions it clearly does not. Let us start with a configuration on $\mathbf{Z}_N^d$ and scale space by N and time by N^2. The generator becomes $N^2\mathcal{A}$ and the particles can be visualized as moving in a lattice imbedded in the unit torus $\mathbf{T}^d$, with spacing $\frac{1}{N}$, and becoming dense as $N \to \infty$.

Let J be a smooth function on $\mathbf{T}^d$. We consider the functional

$$\xi(t) = \frac{1}{N^d}\sum_x J(\frac{x}{N})\eta(t,x)$$

and we can write

$$\xi(t) - \xi(0) = \int_0^t V_N(\eta(s))ds + M_N(t)$$

where

$$V_N(\eta) = (N^2\mathcal{A}V_J)(\eta) = V_{J_N}(\eta)$$

with

$$\begin{aligned}(J_N)(u) &= N^2\sum\big[J(u+\frac{z}{N}) - J(u)\big]p(z)\\ &\simeq \frac{1}{2}(\Delta_C J)(u)\end{aligned}$$

for $u \in \mathbf{T}^d$. Here Δ_C refers to the Laplacian

$$\sum_{i,j} C_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}$$

with the covariance matrix C given by

$$C_{i,j} = \sum_z z_i z_j p(z)$$

$M_N(t)$ is a martingale and a very elementary calculation yields

$$E\Big\{[M_N(t)]^2\Big\} \le C\,t\,N^{-d}$$

essentially completing the proof in this case. Technically the empirical distribution $\nu_N(t)$ is viewed as a measure on $\mathbf{T}^d$ and $\nu_N(\cdot)$ is viewed as a stochastic process with values in the space $\mathcal{M}(\mathbf{T}^d)$ of nonnegative measures on $\mathbf{T}^d$. In the limit it lives on the set of weak solutions of the heat equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\Delta_C\,\rho$$

and the uniqueness of such weak soultions for given initial density establishes the validity of the hydrodynamic limit.

Let us make the problem slightly more complicated by adding a small bias. Let $q(z)$ be an odd function with $q(-z) = -q(z)$ and we will modify the problem by making π depend on N in the form

$$\pi_N(z) = \pi(z) + \frac{1}{N} q(z)$$

Assuming that q is nonzero only when π is so, π_N will be an admissible transition probability for large enough N. A calculation yields that in the slightly modified model referred to as weakly asymmetric simple exclusion model V_N is given by

$$V_N(\eta) \simeq V_{J_N}(\eta) + \frac{1}{N^d} \sum_x \eta(x)(1 - \eta(x)) < m \,, \nabla J(x) >$$

with

$$m = \sum_z z\, q(z)$$

If one thinks of $\rho(t\,, u)$ as the density of particles at the (macroscopic) time t and space u the first term clearly wants to have the limit

$$\int_{\mathbf{T}^d} \frac{1}{2} (\Delta_C J)(u) \rho(t\,, u) du$$

It is not so clear what to do with the second term. The 'invariant' measures in this model are the Bernoulli measures with various densities ρ and the 'averaged' version of the second term should be

$$\int_{\mathbf{T}^d} < m \,, (\nabla J)(u) > \rho(t\,, u)(1 - \rho(t\,, u)) du$$

Replacing the linear heat equation by the nonlinear equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{2} \Delta_C \rho - \nabla \cdot m \rho (1 - \rho)$$

This requires justification that will be the content of our next lecture.

Let us now turn to the case where p has mean zero but is not symmetric. In this case

$$V_N(\eta) = N^{2-d} \sum_{x,y} \eta(x)(1 - \eta(y)) p(y - x)[J(\frac{y}{N}) - J(\frac{x}{N})]$$

and we get stuck at this point. If p is symmetric, as we saw, we gain a factor of N^{-2}. Otherwise the gain is only a factor of N^{-1} which is not enough.

We seem to end up with

$$\begin{aligned} &N^{-d} \sum_x \sum_y \eta(x) < \frac{1}{2} [(\nabla J)(\frac{x}{N}) + (\nabla J)(\frac{y}{N})] \,, N(1 - \eta(y))(y - x) p(y - x) > \\ &= \frac{1}{2N^d} \sum_x (\nabla J)(\frac{x}{N}) N \mathbf{\Psi}_x \end{aligned}$$

where

$$
\begin{aligned}
\boldsymbol{\Psi}_x &= [\eta(x)\sum_z(1-\eta(x+z))z\,p(z) + (1-\eta(x))\sum_z \eta(x-z)z\,p(z)] \\
&= [-\eta(x)\sum_z \eta(x+z)z\,p(z) + (1-\eta(x))\sum_z \eta(x-z)z\,p(z)] \\
&= [\sum_z \eta(x-z)z\,p(z) - \eta(x)\sum_z(\eta(x+z)+\eta(x-z))z\,p(z)] \\
&= \tau_x \boldsymbol{\Psi}_0
\end{aligned}
$$

with τ_x being the shift by x. In the symmetric case, the second sum is zero and $\boldsymbol{\Psi}_0$ can then be written as a 'gradient'

$$\boldsymbol{\Psi}_0 = \sum_j \tau_{e_j}\xi_j - \xi_j$$

where τ_{e_j} are shifts in the coordinate directions. This allows us to do summation by parts and gain a factor of N^{-1}. When this is not the case, we have a 'nongradient' model and the hydrodynamic limit can no longer be established by simple averaging.

4.3 Symmetric Simple Exclusion.

We will now begin the analysis of the symmetric simple exclusion process. We have a probability distribution $\pi(z)$, on $\mathbb{Z}^d$ that is symmetric and compactly supported. We will also assume that its support generates entire $\mathbb{Z}^d$. The covariance matrix C is defined by

$$\langle C\ell, \ell\rangle = \sum_z \langle z, \ell\rangle^2 \pi(z)$$

and is positive definite. The simple exclusion process is defined by the generator

$$(\mathcal{L}f)(\eta) = \sum_{x,y} \pi(y-x)\eta(x)(1-\eta(y))[f(\eta^{x,y}) - f(\eta)]$$

Here the state space consists of Ω consisting of $\eta : \mathbb{Z}_N^d \to \{0,1\}$. η represents a configuration of particles with at most one particle per site. Let us recall that the convention is that $\eta(x) = 1$ if there is particle at x and 0 otherwise and that $\eta^{x,y}$ represents the configuration obtained by exchanging the situation at sites x and y.

$$\eta^{x,y}(z) = \begin{cases} \eta(y) & if\ z = x \\ \eta(x) & if\ z = y \\ \eta(z) & otherwise \end{cases}$$

The initial configuration is a state $\eta_0 = \{\eta(0,x)\}$ and we assume that for some density $\rho_0(u)$ on the torus $\mathcal{T}^d$, $0 \le \rho_0(u) \le 1$, we have

$$\lim_{N\to\infty} \frac{1}{N^d} \sum_{x\in\mathbb{Z}_N^d} J(\frac{x}{N})\eta(0,x) \to \int_{\mathcal{T}^d} J(u)\rho_0(u)du$$

for every continuous function $J : \mathcal{T}^d \to \mathbb{R}$. $\rho_0(\cdot)$ should be thought of as the macroscopic density profile. We speed time up by factor of N^2 and let $P_N = P_{N,\eta_0}$ be the probability measure on the space $D[[0,T];\Omega_N]$. We can map the configuration η to the measure $\lambda_N(du) = \frac{1}{N^d}\sum_x \delta_{\frac{x}{N}}$ on $\mathcal{T}^d$. This induces a measure Q_N on the space $D[[0,T];\mathcal{M}(\mathcal{T}^d)]$. We will prove the following theorem which is quite elementary.

Theorem 4.3.1. *As $N \to \infty$, the distributions Q_N converge weakly to the degenerate distribution concentrated on the trajectory $\rho(t,u)$ which is the unique solution of the heat equation*

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\sum_{i,j=1}^{d} \Delta_C\, \rho$$

with the initial condition $\rho(0,u) = \rho_0(u)$.

Proof. Consider the function

$$F_J(\eta) = \frac{1}{N^d}\sum_{x\in\mathbb{Z}_N^d} J(\frac{x}{N})\eta(x)$$

We can compute

$$\begin{aligned}
(N^{2-d}\mathcal{L}_N F_J)(\eta) &= N^{2-d}\sum_{x,y}[J(\frac{y}{N}) - J(\frac{x}{N})]\eta(x)(1-\eta(y))\pi(y-x)\\
&= \frac{N^{2-d}}{2}\sum_{x,y}[\eta(x)(1-\eta(y)) - \eta(y)(1-\eta(x))][J(\frac{y}{N}) - J(\frac{x}{N})]\pi(y-x)\\
&= \frac{N^{2-d}}{2}\sum_{x,y}[\eta(x) - \eta(y)][J(\frac{y}{N}) - J(\frac{x}{N})]\pi(y-x)\\
&= \frac{N^{2-d}}{2}\sum_{x,z}[\eta(x) - \eta(x+z)][J(\frac{x+z}{N}) - J(\frac{x}{N})]\pi(z)\\
&= \frac{N^{2-d}}{2}\sum_{x,z}\eta(x)[J(\frac{x+z}{N}) - 2J(\frac{x}{N}) + J(\frac{x-z}{N}]\pi(z)\\
&\simeq \frac{1}{2N^d}\sum_{x}\eta(x)(\Delta_C J)(\frac{x}{N})\\
&= \frac{1}{2}\int(\Delta_C J)(u)\lambda_N(du)
\end{aligned}$$

We now establish two things. The sequence Q_N is compact as probability distributions on $D[[0,T];\Omega]$. Any limit Q is concentrated on the set of weak solutions of the heat equation with the correct initial condition. Since there is only one solution, we have weak convergence to the degenerate distribution concentrated at that solution, as claimed.

We will show compactness in $D[[0,T];\mathcal{M}(\mathcal{T}^d)]$. The topology on $\mathcal{M}(\mathcal{T}^d)$ is weak convergence. The space is compact. If we use a continuous test function J, the jump sizes are at most $\sup_{|x-y|\le \frac{C}{N}} |J(x)-J(y)|$, where C is the size of the support of $\pi(\cdot)$. It is therefore sufficient to get a uniform estimate on

$$\theta_{N,J}(\delta,\epsilon) = \sup_{N,\eta} P_{N,\eta}\big[\sup_{0\le t\le\delta} |F_J(t)-F_J(0)| \ge \epsilon\big]$$

and show that for any smooth J and $\epsilon > 0$,

$$\lim_{\delta\to 0}\sup_N \theta_{N,J}(\delta,\epsilon) \to 0$$

From our computation of $(N^2\mathcal{L}_N F)(\eta)$ it follows that if J has two bounded derivatives, then

$$\sup_\eta \sup_N |(\mathcal{L}_N F_J)(\eta)| \le C(J)$$

From the generator $N^2\mathcal{L}_N$. we occlude that

$$F_J(\eta(t)) - F_J(\eta(0)) - N^2\int_0^t (\mathcal{L}_N F_J)(\eta(s))ds = M_{J,N}(t)$$

is a Martingale. $N^2|(\mathcal{L}_N F_J)(\eta)| \le C(J)$ and the quadratic variation of the martingale $M_{J,N}(t)$ is easily estimated. The jumps are of size $\frac{C(J)}{N^{d+1}}$. The total jump rate is at most N^{d+2}. The quadratic variation is then bounded by

$$[C(J)]^2 N^{-2d-2} N^{d+2} = C(J)N^{-d}$$

This is sufficient to prove the compactness of Q_N and that any limit point Q will have the property that for any J

$$\int_{\mathcal{T}^d} J(u)\lambda(t,u)du - \int_{\mathcal{T}^d} J(u)\lambda(u)du - \frac{1}{2}\int_0^t \int_{\mathcal{T}^d} (\Delta_C\, J)(u)\lambda(s,du)ds \equiv 0$$

a.e Q. □

Remark 4.3.2. *The symmetry of $\pi(\cdot)$ played a crucial part. $\eta(x)(1-\eta(y)) - \eta(y)(1-\eta(x)) = \eta(x)-\eta(y)$ cancelled the nonlinearity. If we assumed only $\sum_z z\pi(z) = 0$, we will not have the second difference and $N^2(\mathcal{L}_N F_j)(\eta)$ will be of size N.*

4.4 Large deviations and weak asymmetry.

Now that we have established the Law of large numbers, let us investigate the large deviation properties. We want to obtain a rate function $I(\lambda(\cdot))$ on $D[[0,T];\mathcal{M}(\mathcal{T}^d)]$ and prove a large deviation result for Q_N with this rate function. Lower bound requires a tilting argument and upper bound involves estimates via some exponential martingales. First we need to consider a class of tilts and then optimize over them. The tilts depend on the choice of a function $q(z)$ that is odd i.e. $q(z) = -q(z)$, and $q(z) = 0$ unless $\pi(z) > 0$. Then we perturb $\pi(z)$ by $\pi(z) + \frac{1}{N}q(z)$ that introduces a weak asymmetry. The choice of $q(\cdot) = q(s,u,\cdot)$ can depend on u and s. The generator is modified accordingly

$$\begin{aligned}(N^2\mathcal{L}_{N,q}F_J)(\eta) &= N^{2-d}\sum_{x,y}[J(\frac{y}{N}) - J(\frac{x}{N})]\eta(x)(1-\eta(y))[\pi(y-x) + \frac{1}{N}q(s,\frac{x}{n},z)] \\ &\simeq \frac{1}{2N^d}\sum_x \eta(x)\sum_{i,j} C_{i,j}(D_iD_jJ)(\frac{x}{N}) \\ &\quad + \frac{1}{N^d}\langle\sum_z q(s,\frac{x}{n},z)\eta(x)(1-\eta(x+z)),(\nabla J)(\frac{x}{N}\rangle\end{aligned}$$

where the first term can be replaced as before by

$$\frac{1}{2}\int_{\mathcal{T}^d}(\Delta_C\, J)(u)\lambda(du)$$

It is not clear what to do with the second term. In the limit when $\lambda(du)$ becomes $\rho(u)du$ we would like this term to be

$$\int_{\mathcal{T}^d}\rho(u)(1-\rho(u))\langle m(s,u),(\nabla J)(u)\rangle du$$

where

$$m(s,u) = \sum_z z\,q(s,u,z)$$

because we expect locally the statistics to reflect the Bernoulli distribution with the correct density. If we can accomplish it, we will establish the following theorem.

Theorem 4.4.1. *The sequence of measures $Q_{N,q}$ converges as $N\to\infty$ to the distribution concentrated on the single trajectory that is the weak solution of*

$$\frac{\partial\rho(t,u)}{\partial t} = \frac{1}{2}\Delta_C\rho(t,u) - \nabla\cdot(\rho(t,u)(1-\rho(t,u))m(t,u))$$

with $\rho(0,u) = \rho_0(u)$.

But this requires justification. The route is complicated. There are various approximations that are needed. There are several measures. P_N, Q_N and the perturbed ones $P_{N,q}$ and $Q_{N,q}$. Computations with P_N are easier because it is a reversible Markov process. Direct computations with $P_{N,q}$ are harder. Even while examining P_N it is easier if we start in equilibrium, i.e. with a reversible invariant measure, uniform distribution of ρN^d particles among the N^d sites. One can use Feynman-Kac formula and some variational methods to obtain estimates. We can then transfer the estimates from P_N^{eq} to P_N and $P_{N,q}$ using entropy inequality.

We will deal with averages of all kinds. Let us set up some notation. If $f = f(\eta)$ is a local function on the configuration space Ω or Ω_N their averages will be denoted

$$\bar{f}_{\ell,x} = \frac{1}{(2\ell+1)^d} \sum_{y:|y-x|\leq \ell} f(\tau_y \eta)$$

A special case is when $f(\eta) = \eta(0)$, in which case we denote by

$$\bar{\eta}_{\ell,x} = \frac{1}{(2\ell+1)^d} \sum_{y:|y-x|\leq \ell} \eta(y)$$

$\eta(x)$ can be $\eta(s,x)$ representing the configuration at some time s. We denote by $\tau_x \eta$ the configuration $(\tau_x \eta)(y) = \eta(x+y)$. The object we need to estimate is

$$\int_0^T e_N(\epsilon, s) ds$$

where

$$e_N(\epsilon, s) = E^{P_{N,q}}\left[\left| \frac{1}{N^d} \sum J(\frac{x}{N})[f_x(\eta(s)) - \hat{f}(\bar{\eta}_{N\epsilon,s,x})] \right| \right]$$

and

$$\hat{f}(\rho) = E^{\mu_\rho}[f(\eta)]$$

the expectation of the local function f with respect to the product Bernoulli measure with density ρ.

Let $\mu_N(s, d\eta)$ be the marginal distribution on Ω_N of the configuration $\eta(s,\cdot)$ at time s and $\bar{\mu}_N$ its space and time average. More precisely

$$\int f(\eta) d\bar{\mu}_N(\eta) = \frac{1}{T} \int_0^T \int_{\Omega_N} \left[\frac{1}{N^d} \sum_x f(\tau_x \eta) \right] \mu_N(s, d\eta) ds$$

We need the following theorem

Theorem 4.4.2.

$$\lim_{\epsilon \to 0} \limsup_{N \to \infty} \int_0^T e_N(\epsilon, s) ds = 0$$

This is proved in two steps.

Lemma 4.4.3.
$$\lim_{k\to\infty}\limsup_{N\to\infty}\left|\bar{f}_k(\eta)-\hat{f}(\bar{\eta}(k))|\right] = 0$$

Lemma 4.4.4.
$$\limsup_{\substack{\epsilon\to 0\\ k\to\infty}}\limsup_{N\to\infty} E^{\bar{\mu}_N}\left[|\bar{\eta}_{N\epsilon}-\bar{\eta}_k| = 0\right.$$

We note the following.

- We can always replace for fixed k and large N, the sum
$$\frac{1}{N^d}\sum_x J(\frac{x}{N})f(\tau_x\eta)$$
by
$$\frac{1}{N^d}\sum_x f(\tau_x\eta)\frac{1}{(2k+1)^d}\sum_{y:|y-x|\le k} J(\frac{y}{N})$$
or
$$\frac{1}{N^d}\sum_x J(\frac{x}{N})\bar{f}_{k,x}$$

- Lemma 4.4.3 allows us to replace this by
$$\frac{1}{N^d}\sum_x J(\frac{x}{N})\hat{f}(\bar{\eta}_{k,x})$$

- Lemma 4.4.4 allows us to replace $\bar{\eta}_{k,x}$ with large k by $\bar{\eta}_{N\epsilon,x}$ with a small ϵ.

- This allows us to replace $\eta(x)(1-\eta(x))$ by $\bar{\eta}_{N\epsilon,x}(1-\bar{\eta}_{N\epsilon,x})$, which is sufficient to prove Theorem 4.4.2

We now concentrate on the proof of the two lemmas. They depend on the following observations.

- The function $f(a,b)=(\sqrt{a}-\sqrt{b})^2$ is a convex function of $(a,b)\in\mathbb{R}_+^2$. It is checked by computing the Hessian
$$\frac{1}{2}\begin{pmatrix} b^{\frac{1}{2}}a^{-\frac{3}{2}} & -(ab)^{-\frac{1}{2}} \\ -(ab)^{-\frac{1}{2}} & a^{\frac{1}{2}}b^{-\frac{3}{2}}\end{pmatrix}$$
of f which is seen to be positive semidefinite.

- The invariant distributions fot $\mathcal{A}$ are all reversible and form a convex set with extreme points $w_{N,k}$ which are uniform distributions over $\binom{2^{N^d}}{k}$ configurations in Ω_N with a given number $k = \sum_{x\in\mathbb{Z}_N^d} \eta(x)$ of occupied sites.

  The Dirichlet form corresponding to the operator $\mathcal{A}$ is given by

$$\frac{1}{2}\sum_{\eta}\sum_{x,y}[f(\eta^{x,y}) - f(\eta)]^2\pi(y-x)w_{N,k}(\eta)$$

  where $w_{N,k}(\eta)$ is any reversible invariant distribution.

- If $\pi(\cdot)$ is irreducible, then they are all equivalent to

$$\sum_{\eta}\sum_{\substack{x,y\\|x-y|=1}}[f(\eta^{x,y}) - f(\eta)]^2 w_{N,k}(\eta)$$

- If $\mu_N = \{\mu_N(\eta)\}$ is a probability distribution on Ω_N its Dirichlet form is defined by

$$\mathcal{D}(\sqrt{\mu_N}) = \frac{1}{2}\sum_{\eta}\sum_{|x-y|=1}(\sqrt{\mu_N(\eta^{x,y})} - \sqrt{\mu_N(\eta)})^2$$

- Let us denote by $\alpha_N(k) = \mu_N\{\sum_{x\in\mathbb{Z}_N^d}\eta(x) = k\}$. Then the invariant distribution that corresponds to μ_N is the convex combination $w_N = \sum_k \alpha_N(k)\, w_{N,k}$. If we write $\mu_N(\eta) = f(\eta)w_N(\eta)$ then $\sqrt{f}$ is in $L_2(w_N)$ with $\|\sqrt{f}\|_{L_2(w_N)} = 1$ and $\mathcal{D}(\sqrt{f}) = \mathcal{D}(\sqrt{\mu_N})$.

- An estimate of the form $\mathcal{D}(\sqrt{\mu_N}) \le CN^{d-2}$ has many consequences. If $\mu_N(s)$ are such that $\int_0^T D(\sqrt{\mu_N(s)})ds \le CN^{d-2}$, then the average $\bar{\mu}_N$ over space and time, defined by,

$$\bar{\mu}_N = \frac{1}{T}\frac{1}{N^d}\int_0^T\sum_x \tau_x\bar{\mu}_N(s)ds$$

  satisfies

$$\mathcal{D}(\sqrt{\bar{\mu}_N}) \le \frac{C}{T}N^{d-2}$$

  Since there is spacial homogeneity the quantity

$$\sup_x\sum_{\eta}\sum_{y:|x-y|=1}(\sqrt{\bar{\mu}_N(\eta^{x,y})} - \sqrt{\bar{\mu}_N(\eta)})^2 \le \frac{C}{T}N^{-2}$$

- If we restrict the distribution to a block $\mathbf{B}_k$ of size k, the marginal distribution has the Dirichlet form from bonds $(x,y) : |x-y| = 1$ internal to $\mathbf{B}_k$ which is at most $(2k)^d\frac{C}{T}N^{-2}$. As $N \to \infty$ the Dirichlet form goes to 0. The limiting distribution is therefore permutation invariant. Hence it is uniform over all possible choices of locations for the total number of particles in the block. This shows that

$$\limsup_{k\to\infty}\limsup_{N\to\infty} E^{\bar\mu_N}\left[\bar f_k - \hat f(\bar\eta_k)\right] = 0$$

This is precisely lemma 4.4.3.

- We want to estimate $\sum_\eta(\sqrt{\mu_N(\eta^{x,y})}-\sqrt{\mu_N(\eta)})^2$ when $|x-y| = \ell$ instead of $|x-y| = 1$. Any interchange of x, y at a distance ℓ can be achieved by 2ℓ successive interchanges of nearest neighbors and the simple inequality

  $$(a_1 + \cdots + a_\ell)^2 \le \ell \sum_{j=1}^{\ell} a_j^2$$

  allows us to estimate

  $$\sum_\eta(\sqrt{\mu_N(\eta^{x,x+z})} - \sqrt{\mu_N(\eta)})^2 \le |z|^2 \frac{C}{T} N^{-2}$$

  In particular if $|z| \le N\epsilon$ then

  $$\sum_\eta(\sqrt{\mu_N(\eta^{x,x+z})} - \sqrt{\mu_N(\eta)})^2 \le |z|^2 \frac{C}{T} N^{-2} \le \epsilon^2 \frac{C}{T}$$

- This shows that if two microscopically large blocks are macroscopically close then there empirical densities are close with probability nearly 1. This implies lemma 4.4.4.

All we need to do now is control the Dirichlet form. Our generator is of the form

$$\mathcal{A}_{N,q} = \mathcal{A} + \frac{1}{N}\mathcal{S}_{N,q}$$

the sum of the symmetric part $\mathcal{A}$ and the weak skew symmetric perturbation $\mathcal{S}_{N,q}$. The entropy is defined by

$$\mathcal{H}(\mu) = \sum_\eta \mu(\eta)\log\frac{\mu(\eta)}{\alpha(\eta)}$$

where $\alpha(\eta) = c = c(N,\rho) = \binom{N^d}{k_N}^{-1}$ is the uniform distribution of $k_N = \rho N^d$ particles on Z_N^d. It is invariant for the evolution under $\mathcal{A}$.

Let us note that μ_N^t evolves according to the forward equation $\frac{d\mu_N^t}{dt} = N^2 \mathcal{A}_{N,q}^* \mu_N$. Therefor for $H_N(t) = \mathcal{H}(\mu_N^t)$ we have

$$\begin{aligned}
\frac{dH_N(t)}{dt} &= \frac{d}{dt} \sum_\eta \log \frac{\mu_N^t(\eta)}{c} d\mu_N^t(\eta) \\
&= N^2 \sum_\eta \log \frac{\mu_N^t(\eta)}{c} \mathcal{A}_{N,q}^* \mu_N^t + \sum_\eta \frac{d}{dt} \mu^t(\eta) \\
&= N^2 \sum_\eta (\mathcal{L}_{N,q} \log \mu_N^t(\eta)) \mu_N^t(\eta) \\
&= N^2 \sum_\eta \sum_{x,y} [\pi(y-x) + \frac{1}{N} q(t, \frac{x}{N}, y-x)] \log \frac{\mu_N^t(\eta^{x,y})}{\mu_N^t(\eta)} \eta(x)(1-\eta(y)) \mu_N^t(\eta) \\
&= N^2 \sum_\eta \sum_{x,y} \pi(y-x)[1 + \frac{q(t, \frac{x}{N}, y-x)}{N\pi(y-x)}] \log \frac{\mu_N^t(\eta^{x,y})}{\mu_N^t(\eta)} \eta(x)(1-\eta(y)) \mu_N^t(\eta)
\end{aligned}$$

Denoting by $c_N = c_N(t,u,y-x) = 1 + \frac{q(t,u,y-x)}{N\pi(y-x)}$, and using the inequality

$$x \log y \le 2[x \log x - x + 1] + 2[\sqrt{y} - 1]$$

(verify $\sup_x [x \log y - 2(x \log x - x + 1)] = 2(\sqrt{y} - 1)$), we obtain

$$\begin{aligned}
\frac{dH_N(t)}{dt} &\le 2N^2 \sum_\eta \sum_{x,y} \pi(y-x)[c_N \log c_N - c_N + 1] \mu_N^t(\eta) \\
&\quad + 2N^2 \sum_\eta \sum_{x,y} \pi(y-x)[\sqrt{\mu_N^t(\eta^{x,y})} - \sqrt{\mu_N^t(\eta)}] \mu_N^t(\eta)
\end{aligned}$$

$$2N^2(c_N \log c_N - c_N + 1) \simeq \left[\frac{q(t, \frac{x}{N}, y-x)}{\pi(y-x)} \right]^2$$

We can also change $\eta \to \eta^{x,y}$ in the summation. It just maps Ω_N onto itself. Assuming $|q(t,u,z)| \le C\pi(z)$, we end up with

$$\frac{dH_N(t)}{dt} \le CN^d - N^2 \sum_\eta \sum_{x,y} \pi(y-x)(\sqrt{\mu_N^t(\eta^{x,y})} - \sqrt{\mu_N^t(\eta)})^2$$

If the support of $\pi(\cdot)$ generates $\mathbb{Z}^d$ it is routine to estimate

$$\mathcal{D}(f) = \sum_{|x-y|=1} [f(\eta^{x,y} - f(\eta)]^2 \le C \sum_{x,y} \pi(y-x)[f(\eta^{x,y}) - f(\eta)]^2$$

This leads to the estimate

$$H_N(T) - H_N(0) \le CTN^d - \frac{N^2}{C}\int_0^T \mathcal{D}\big(\sqrt{\mu_N^t}\big)dt$$

Since $H_N(T) \ge 0$ and $H_N(0) \le CN^d$, it follows that

$$\int_0^T \mathcal{D}\big(\sqrt{\mu_N^t}\big)dt \le C(T)N^{d-2}$$

4.5 Super-exponential estimates.

A consequence of the estimates of the previous section is the following theorem. Let

$$\begin{aligned}
\mathcal{M}_{N,C} &= \Big\{\mu_N(\cdot) : \mathcal{D}(\sqrt{\mu_N}) \le CN^{d-2}\Big\} \\
e_{N,k}(f,\eta) &= \frac{1}{N^d}\sum_{x\in\mathbb{Z}_N^d}\big|\frac{1}{(2k+1)^d}\sum_{|y-x|\le k} f(\tau_y\eta)) - \hat{f}(\bar{\eta}_{x,k})\big| \\
d_{N,k,\epsilon}(\eta) &= \frac{1}{N^d}\sum_{x\in\mathbb{Z}_N^d}|\bar{\eta}_{x,k} - \bar{\eta}_{x,N\epsilon}|
\end{aligned}$$

Then

Theorem 4.5.1. *For any $C < \infty$ and local function f,*

$$\lim_{k\to\infty}\limsup_{N\to\infty}\sup_{\mu_N\in\mathcal{M}_{N,C}} E^{\mu_N}\big[e_{N,k}(f,\eta)\big] = 0$$

$$\lim_{\substack{k\to\infty\\ \epsilon\to 0}}\limsup_{N\to\infty}\sup_{\mu_N\in\mathcal{M}_{N,C}} E^{\mu_N}\big[d_{N,k,\epsilon}(\eta)\big] = 0$$

Let $\{P_x\}$ $x \in \mathcal{X}$ be a Markov family on the finite state space $\mathcal{X}$ i.e. measures on $D[[0,\infty),\mathcal{X}]$ corresponding to a generator

$$(Af)(x) = \sum_y [f(y) - f(x)]q(x,y)$$

Assume that $\mu(x)$ is an invariant probability distribution for the Markov process. Let us assume that the process is reversible with respect to μ, i.e. A is self adjoint in $L_2(\mu)$. This is the same as $q(x,y)\mu(x) = q(y,x)\mu(y)$. The Dirichlet form is given by

$$-\langle Af, f\rangle_\mu = -\sum_{x,y}[f(y) - f(x)]f(x)q(x,y)\mu(x) = \frac{1}{2}\sum_{x,y}[f(y) - f(x)]^2 q(x,y)\mu(x)$$

If the process is not reversible then the Dirichlet form corresponds to $\frac{1}{2}(A + A^*)$.

The Feynman-Kac formula says that for any $V : \mathcal{X} \to \mathbb{R}$,

$$u(t, x) = E^{P_x}[\exp[\int_0^t V(x(t))dt]f(x(t))]$$

is the solution of

$$\frac{du(t, x)}{dt} = (Au)(t, x) + V(x)u(t, x);\ \ u(0, x) = f(x)$$

It follows that

$$\frac{d\|u(t, \cdot)\|_2^2}{dt} = \langle (A + A^*)u(t) + 2Vu(t), u(t)\rangle_\mu = 2\langle u(t)V, u(t)\rangle - 2\mathcal{D}(u(t))] \leq 2\lambda_A(V)\|u(t)\|_2^2$$

providing the estimate

$$\|u(t, \cdot)\|_2^2 \leq \exp[2t\lambda_A(V)]\|f\|_2^2$$

where

$$\lambda_A(V) = \sup_{u:\|u\|_2=1} \big[\langle uV, u\rangle - \mathcal{D}(u)\big]$$

Taking $f = 1$ and using Schwartz's inequality

$$E^{P_\mu}[\exp[\int_0^t V(x(s))ds]] = \|u(t)\|_1 \leq \exp[t\lambda_A(V)]$$

(There are no constants!)

The following lemma is useful and replaces L_p, L_q duality of Hölder's inequality with the entropy duality between $x \log x$ and e^x.

Lemma 4.5.2. *Let $Q << P$ with $H(Q; P) = \int \frac{dQ}{dP} \log \frac{dQ}{dP} dP < \infty$. Then for any bounded measurable function $g(x)$*

$$E^Q[g(x)] \leq H(Q; P) + \log \int e^{g(x)}dP \tag{4.3}$$

For any set A,

$$Q(A) \leq \frac{H(Q; P) + 1}{\log \frac{1}{P(A)}} \tag{4.4}$$

Proof. The inequality

$$ab \leq b \log b - b + e^a$$

implies, with $f(x) = \frac{dQ}{dP}$, for any $c > 0$, (replacing b by cb and dividing by c),

$$f(x)g(x) \leq f(x) \log[cf(x)] - f(x) + \frac{1}{c}e^{g(x)}$$

Integrating with respect to P we obtain,

$$\int g\,dQ \leq \log c + H(Q;P) - 1 + \frac{1}{c}\int e^{g(x)}dP$$

Pick $c = \int e^{g(x)}dP$. We then get

$$\int g\,dQ \leq H(Q;P) + \log \int e^{g(x)}dP$$

Take $g(x) = k\mathbf{1}_A(x)$ and $k = \log \frac{1}{P(A)}$. Then

$$\begin{aligned}
Q(A) &\leq \frac{1}{k}[H(Q;P) + \log \int e^{k\mathbf{1}_A(x) + \mathbf{1}_{A^c}(x)}dP] \\
&= \frac{1}{k}[H(Q;P) + \log[e^k P(A) + P(A^c)]] \\
&= \frac{1}{k}[H(Q;P) + \log 2] \leq \frac{H(Q;P)+1}{\log \frac{1}{P(A)}}
\end{aligned}$$

□

The following theorem is now an easy consequence of theorem 4.5.1

Theorem 4.5.3. *Let $\eta \in \Omega_N$ be arbitrary. Then for any $\delta > 0$ and local function f,*

$$\limsup_{k\to\infty}\limsup_{N\to\infty}\frac{1}{N^d}\sup_{\eta}\log P^{\eta}[\int_0^T e_{N,k}(f,\eta(t))dt \geq \delta] = -\infty$$

$$\limsup_{\substack{k\to\infty\\ \epsilon\to 0}}\limsup_{N\to\infty}\frac{1}{N^d}\sup_{\eta}\log P^{\eta}[\int_0^T d_{N,k,\epsilon}(\eta(t))dt \geq \delta] = -\infty$$

Proof. We start by estimating

$$\begin{aligned}
&\limsup_{N\to\infty}\frac{1}{N^d}\log E^{P_\mu}\big[\exp[N^d\int_0^T V(\eta(s))ds]\big] \\
&\qquad\leq \limsup_{N\to\infty}\frac{T}{N^d}\sup_{\|u\|_2=1}[N^d\langle Vu,u\rangle - N^2\mathcal{D}(u)] \\
&\qquad= T\limsup_{N\to\infty}\sup_{\|u\|_2=1}[\langle Vu,u\rangle - N^{2-d}\mathcal{D}(u)]
\end{aligned}$$

If $V \geq 0$ is bounded by C, then the supremum can be limited to u with $\mathcal{D}(u) \leq CN^{d-2}$. Taking $\mu_N(\eta) = 2^{-N^d}u^2(\eta)$, if for some V, if we can control

$$\lambda(\sigma V) = \sup_{\|u\|_2=1}[\sigma\langle Vu,u\rangle - N^{2-d}\mathcal{D}(u)]$$

then we can get estimates under P_μ. But each point carries a mass of 2^{-N^d}. If we have super exponential estimates under P_μ we have them uniformly under P_η for $\eta \in \Omega_N$. We can take $V(\eta) = e_{N,k}(f, \eta)$ or $V(\eta) = d_{N,k,\epsilon}(\eta)$ and use Tchebychev's inequality on

$$E^{P_\mu}[\exp N^d[\sigma \int_0^T V(\eta(s))ds]]$$

to get

$$\limsup_{N\to\infty} \frac{1}{N^d} \log P_\mu[\int_0^T V(x(s))ds \geq \delta] \leq [-\sigma\delta + \lambda(\sigma V)]$$

If $\lambda(\sigma V) \to 0$ as $k \to \infty$ (and $\epsilon \to 0$) for every $\sigma > 0$ we have our super exponential estimates. □

Weak asymmetry introduces a perturbation $\pi(y-x) \to \pi(y-x) + \frac{1}{N}q(t, \frac{x}{N}, y-x)$. Assuming $\frac{q(t,u,z)}{\pi(z)}$ is bounded the relative entropy of this perturbation is easily estimated to be

$$N^2 \int_0^T \sum_{\eta,x,y} (c_N \log c_N - c_N + 1)\pi(y-x)\eta(x)(1-\eta(x))\mu_N(t,\eta) \leq CN^d$$

Therefore for the perturbation Q_N, uniformly over all initial configuration

$$\limsup_{k\to\infty} \limsup_{N\to\infty} \sup_\eta Q_N^\eta[\int_0^T e_{N,k}(f, \eta(t))dt \geq \delta] = -\infty$$

$$\limsup_{\substack{k\to\infty\\ \epsilon\to 0}} \limsup_{N\to\infty} \sup_\eta Q_N^\eta[\int_0^T d_{N,k,\epsilon}(\eta(t))dt \geq \delta] = -\infty$$

Now we are ready to establish the large deviation lower bound.

Theorem 4.5.4. *Let $\rho(t,x)$ be a weak solution of*

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\Delta_C\, \rho - \nabla \cdot b(t,u)\rho(t,u)(1-\rho(t,u)) = 0;\ \ \rho(0,u) = \rho_0(u) \tag{4.5}$$

with a bounded continuous $b(t,u)$. Then for initial condition η_0 compatible with ρ_0 and for any neighborhood G of $\rho(\cdot)$ in $C[[0,T]; \mathcal{M}(\mathcal{T}^d)]$

$$\liminf_{N\to\infty} \frac{1}{N^d} \log P_N^{\eta_0}[\lambda(\cdot) \in G] \geq -\frac{1}{2}\int_0^T \int_{\mathcal{T}^d} \langle C^{-1}b, b\rangle \rho(t,u)(1-\rho(t,u))dtdu$$

In particular the lower bound for the large deviation can be

$$\inf_{b\in \mathbf{B}(\rho)} \frac{1}{2}\int_0^T \int_{\mathcal{T}^d} \langle C^{-1}b, b\rangle \rho(t,u)(1-\rho(t,u))dtdu$$

where $\mathbf{B}(\rho)$ *consists of* b *that satisfies (4.5). The infimum is calculated to be*

$$\begin{aligned}
&\frac{1}{2}\int_0^T \|\rho_t - \frac{1}{2}\sum_{i,j} C_{i,j}D_iD_j\rho\|^2_{-1,C,\rho(1-\rho)}dt \\
&\quad = \sup_{J(\cdot,\cdot)}\Big[\int_{\mathcal{T}^d} J(T,u)\rho(T,u)du - \int_{\mathcal{T}^d} J(0,u)\rho(0,u)du \\
&\quad\quad - \frac{1}{2}\int_0^T\int_{\mathcal{T}^d}[J_t + \frac{1}{2}\sum_{i,j} C_{i,j}D_iD_jJ]\rho(t,x) - \frac{1}{2}\int_0^T \langle C\nabla J, \nabla J\rangle \rho(t,u)(1-\rho(t,u))dtdu\Big]
\end{aligned}$$

Proof. We begin by choosing a perturbation $q(t,u,z)$. We need to choose q so that

$$b(t,u) = \sum_z zq(t,u,z)$$

while minimizing

$$\sum_z \frac{[q(t,u,z)]^2}{\pi(z)}$$

The choice is

$$q(t,u,z) = \langle C^{-1}b(t,u), z\rangle \pi(z)$$

and the minimum is $\langle C^{-1}b, b\rangle$. Here $C = \{C_{i,j}\}$ is the covariance matrix of $\pi(\cdot)$. We next need to prove compactness in $C[[0,T];\mathcal{M}(\mathcal{T}^d)]$ for the perturbed process. We will use repeatedly inequality (4.4). In particular if $-\log P(A) >> N^d$ and $H(Q,P) = O(N^d)$ then $Q(A) << 1$. To prove compactness we will show that a super exponential tightness estimate holds in equilibrium. Then any limit will be concentrated on the weak solutions of (4.5), and we need only prove uniqueness of them. The entropy will then have the correct limit.

□

Actually there is a converse as well.

Theorem 4.5.5. *If* P_N *is a sequence of probability distributions on* $\mathcal{X}$ *such that every sequence* Q_N *with* $H(Q_N;P_N) \le CN$ *is uniformly tight, then* P_N *is super exponentially tight.*

Proof. Consider the set $B_\ell = \cup_N\{Q : H(Q,P_N) \le \ell N\}$. B_ℓ is tight. There is a compact set $K_{\ell,\epsilon}$ such that $Q[K^c_{\ell,\epsilon}] \le \epsilon$ for every $Q \in B_\ell$. Take $\epsilon = \frac{1}{2}$. There is K_ℓ such that $Q(K^c_\ell) = 1 \ge \frac{1}{2}$ implies $H(Q;P_N) \ge CN$. Take Q to be the restriction of P_N to K^c_ℓ. The relative entropy of Q to P_N is easily calculated to be $-\log P_N(K^c_\ell)$, proving $P_N(K^c_\ell) \le \exp[-\ell N]$. □

4.6 Exponential martingales.

For any Markov process with generator $\mathcal{A}$, there are exponential martingales associated with them. They are of the form

$$\exp[f(x(t)) - f(x(0)) - \int_0^t (e^{-f}\mathcal{A}e^f)(x(s))ds]$$

and are quite useful for estimates.

In the case of the simple exclusion process, with

$$f(\eta) = \sum_x J(\frac{x}{N})\eta(x)$$

we obtain the martingales

$$M_{N,J}(t) = \exp[\sum_{x\in\mathbb{Z}_N^d} J(\frac{x}{N})[\eta(t,x) - \eta(0,x)] - \int_0^t A_{N,J}(s)ds]$$

where

$$\begin{aligned}
A_{N,J}(s) &= N^2\sum_{x,y}\pi(y-x)[e^{J(\frac{y}{N})-J(\frac{x}{N})} - 1]\eta(s,x)(1-\eta(s,y))\\
&= \frac{N^2}{2}\sum_{x,y}\pi(y-x)\Big[(e^{J(\frac{y}{N})-J(\frac{x}{N})} - 1)\eta(s,x)(1-\eta(s,y))\\
&\qquad\qquad + (e^{J(\frac{x}{N})-J(\frac{y}{N})} - 1)\eta(s,y)(1-\eta(s,x))\Big]\\
&\simeq \frac{N^2}{2}\sum_{x,y}\pi(y-x)(J(\frac{y}{N}) - J(\frac{x}{N}))(\eta(s,x) - \eta(s,y))\\
&\qquad + \frac{N^2}{4}\sum_{x,y}\pi(y-x)(J(\frac{y}{N}) - J(\frac{x}{N}))^2[\eta(x)(1-\eta(y)) + \eta(y)(1-\eta(x))]\\
&\simeq \frac{1}{2}(\Delta_C J)\eta(x) + \frac{1}{4}\sum_{x\in\mathbb{Z}_N^d}\sum_{i,j=1}^d D_iJ(\frac{x}{N})D_jJ(\frac{x}{N})f_x^{i,j}(\eta)
\end{aligned}$$

where

$$f^{i,j}(\eta) = \sum \pi(z)z_iz_j[\eta_x(1-\eta(x+z)) + \eta(x+z)(1-\eta(x))]$$

The exponential martingale has many uses. The first one is super-exponential tightness.

Lemma 4.6.1. *Let $J(u)$ be a smooth function on $\mathcal{T}^d$. Let P_N^η be the simple exclusion process with initial state η. Then for any $\epsilon > 0$,*

$$\limsup_{\delta\to 0}\limsup_{N\to\infty}\sup_\eta \frac{1}{N^d}\log P_N^\eta\big[\frac{1}{N^d}\sup_{0\le t\le\delta}|\sum_{x\in\mathbb{Z}_N^d} J(\frac{x}{N})[\eta(t,x) - \eta(0,x)]| \ge \epsilon\big] = -\infty$$

Proof.

$$\frac{1}{N^d}\sum_{x\in\mathbb{Z}_N^d} J(\frac{x}{N})[\eta(t,x)-\eta(0,x)] - \int_0^t A_{N,J}(s)ds = M_{N,J}(t)$$

is a martingale, where

$$A_{N,J}(s) = \frac{N^2}{2}\frac{1}{N^d}\sum_{x,z}\pi(z)[J(\frac{x+z}{N}) - 2J(\frac{x}{N}) + J(\frac{x-z}{N})]\eta(s,x)$$

and $|A_{N,J}| \leq C(J)$. It is therefore sufficient to estimate

$$P_N^\eta[\sup_{0\leq s\leq t} M_{N,J}(s) \geq \delta]$$

We have the exponential martingales

$$\exp\left[N^d M_{N,J}(t) - \int_0^t B_{N,J}(s)ds\right]$$

where

$$\begin{aligned} B_{N,J}(s) &= N^2\sum_{x,y}[e^{J(\frac{y}{N})-J(\frac{x}{N})} - 1 - (J(\frac{y}{N}) - J(\frac{x}{N}))]\pi(y-x)\eta(s,x)(1-\eta(s,y)) \\ &\leq C(J)N^d \end{aligned}$$

The usual bound (Doob's inequality) for martingale yields

$$\begin{aligned} &P_N^\eta[\sup_{0\leq t\leq\delta} M_{N,J}(t)\geq\delta] \\ &\qquad\leq P[\sup_{0\leq t\leq\delta}\lambda N^d M_{N,J}(t) - C(\lambda J)N^d t \geq \lambda N^d\epsilon - C(\lambda J)N^d\delta] \\ &\qquad\leq e^{-N^d\lambda\epsilon + C(\lambda J)N^d\delta} \end{aligned}$$

Hence

$$\limsup_{N\to\infty}\frac{1}{N^d}\log P_N^\eta[\sup_{0\leq s\leq t} M_{N,J}(s)\geq\delta] \leq -\lambda\epsilon + C(\lambda J)\delta$$

and

$$\limsup_{\delta\to 0}\limsup_{N\to\infty}\sup_\eta\frac{1}{N^d}\log P_N^\eta[\sup_{0\leq s\leq t} M_{N,J}(s)\geq\delta] \leq -\lambda\epsilon$$

Since $\epsilon > 0$ and $\lambda > 0$ is arbitrary we are done. □

This is actually all we need to prove super exponential tightness.

Lemma 4.6.2. *Let P be a probability distribution on $[D[0,T];\mathcal{X}$ where $\mathcal{X}$ is a Polish space. Let*

$$P[\sup_{t\le s\le t+\delta} d(x(t),x(s))\ge \epsilon|\mathcal{F}_t]\le \psi_P(\delta,\epsilon)$$

and

$$P[d(x(t-0),x(t+0))\ge\theta]=0$$

Then for any integer k and $\beta>0$,

$$P[\sup_{\substack{|s-t|\le\delta\\ 0\le s,t\le T}} d(X(t),X(s))\ge 4\epsilon+\theta]\le k\psi_P(\delta,\epsilon)+e^{\beta T}\beta^k\left[\int_0^\infty e^{-\beta\delta}\psi_P(\delta,\epsilon)d\delta\right]^k$$

In particular if X is compact and $\{P_\alpha\}$ is a family such that

$$\lim_{\delta\to 0}\sup_\alpha \psi_{P_\alpha}(\delta,\epsilon)=0$$

P_α is uniformly tight family.

Proof. Define successive stopping times, $\tau_0=0$,

$$\tau_{j+1}=\inf\{t:t\ge\tau_j, d(X(t),X(\tau_j))\ge\epsilon\}$$

Let $k=\inf\{j:\tau_{j+1}>T\}$ and $\delta^*=\inf_{1\le j\le k}(\tau_j-\tau_{j-1})$. Then if $0\le s,t\le T$ and $|t-s|\le\delta^*$, there can be at most one τ_j between them and hence $d(X(s),X(t))\le 4\epsilon+\theta$. So we need to estimate

$$Q_N[\delta^*\le\delta]=\sum_{j=1}^k Q_N[\tau_j\le\delta]+P[\tau_k<T]$$

We have control on the first term

$$k\psi(\delta,\epsilon)$$

As for the second term we can estimate it by

$$P[\tau_k<T]\le e^T E[e^{-\tau_k}]\le e^T\bar\omega^k$$

where $\bar\omega$ is an upper bound on

$$E^P[e^{-(\tau_{j+1}-\tau_j)}|\mathcal{F}_{\tau_{j-1}}]\le\psi_P(\delta,\epsilon)(1-e^{-\delta})+e^{-\delta}$$

If we pick δ_0 such that $\psi_P(\delta_0,\epsilon)\le\frac{1}{2}$ then

$$\bar\omega\le\frac{1+e^{-\delta_0}}{2}$$

for some positive δ_0. ∎

Lemma 4.6.3. *Given any $\ell < \infty$ there exists a compact set $K_\ell \in D[[0,T];\mathcal{M}(\mathcal{X})]$ such that for any $\epsilon > 0$,*

$$\lim_{N\to\infty} \frac{1}{N^d} \sup_\eta \log P_N^\eta[K_\ell^c] \le -\ell$$

Proof. We will use Theorem 4.5.5. It is sufficient to prove that if $H(Q_N : P_N) \le CN^d$, then Q_N is tight. But since we are dealing with weak topology in $\mathcal{M}(\mathcal{X})$ this is the same as tightness of the processes $X_J(t) = \langle J, \lambda(t)\rangle$ under Q_N. From lemma 4.6.1, we have the estimates

$$\psi(\delta,\epsilon) \le e^{-N^d\lambda\epsilon + C(\lambda J)N^d\delta}$$

that are super exponential. Therefore for any sequence Q_N with $H(Q_N;P_N) \le CN$ we will have uniform estimates on $\psi(\delta,\epsilon)$. This will imply the uniform tightness of Q_N. □

The following lemma allows us to use super exponential estimates in evaluating integrals.

Lemma 4.6.4. *Suppose for a sequence of distributions on some Polish space $\mathcal{X}$ we have the estimates*

$$\limsup_{N\to\infty} \frac{1}{N} \log \int \exp[NF_N(x)]dP_N \le C$$

where F_N is a sequence bounded by C. Suppose for each $\epsilon > 0$, $G_{N,\epsilon}$ is another sequence of continuous functions on $\mathcal{X}$, such that they are all uniformly bounded by C, and for any $\delta > 0$,

$$\limsup_{\epsilon\to 0}\limsup_{N\to\infty} \frac{1}{N} \log P_N[|F_N - G_{N,\epsilon}| \ge \delta] = -\infty$$

Assume further that as $N \to \infty$, $G_{N,\epsilon} \to G_\epsilon$ uniformly on compact sets and $G_\epsilon \to G$ uniformly on compacts as $\epsilon \to 0$. Then if Q_N is such that

$$\limsup_{N\to\infty} \frac{1}{N} H(Q_N;P_N) \le H$$

and $Q_N \to Q$ weakly in $\mathcal{M}(\mathcal{X})$, then

$$E^Q[G(x)] \le C + H$$

Proof.

$$E^{Q_N}[G_{N,\epsilon}(x)] \le \frac{1}{\lambda}\big[H(Q_N;P_N) + \log E^{P_N}[\exp[\lambda G_{N,\epsilon}(x)]]$$

On the other hand, writing

$$\exp[\lambda G_{N,\epsilon}] \le \exp[\lambda F_N] \times \exp[\lambda|G_{N,\epsilon} - F_N|]$$

and using Hölder's inequality

$$\log E^{P_N}[\exp[\lambda G_{N,\epsilon}(x)]] \le \lambda \log E^{P_N}[\exp[F_N(x)]] + (1-\lambda)\log E^{P_N}[\exp[\frac{\lambda}{1-\lambda}|G_{N,\epsilon} - F_N|]]$$

The super exponential estimate implies that for any $0 < \lambda < 1$,

$$\limsup_{N\to\infty} \log E^{P_N}[\exp[\frac{\lambda}{1-\lambda}|G_{N,\epsilon} - F_N|]] = 0$$

Therefore

$$E^Q[G_\epsilon] \le \limsup_{N\to\infty} E^{Q_N}[G_{N,\epsilon}] \le \frac{1}{\lambda}[H + C]$$

We can let $\epsilon \to 0$ and $\lambda \to 1$ to complete the proof. □

Now we need to establish the uniqueness of weak solutions to the problem

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\Delta_C\, \rho - \nabla \cdot b(t,u)\rho(t,u)(1-\rho(t,u))$$

The difference of two solutions $r = \rho - \rho'$ satisfies

$$\begin{aligned}\frac{\partial r}{\partial t} &= \frac{1}{2}\Delta_C\, r - \nabla \cdot b(t,u) r(t,u)[1 - \rho(t,u) - \rho'(t,u)] \\ &= \frac{1}{2}\Delta_C\, r - \nabla \cdot c(t,u) r(t,u)\end{aligned}$$

$$\frac{d}{dt}\|r(t)\|_2^2 = -\frac{1}{2}\int_{\mathcal{T}^d} \langle C\nabla r, \nabla r\rangle du + \int_{\mathcal{T}^d} r\langle c \cdot \nabla r\rangle du \le K\|r\|_2^2$$

If we can establish some a priori regularity on the solutions then Gronwall's inequality will establish uniqueness.

We have the exponential martingales for P_N. With

$$\begin{aligned}F_J = \sum_{x\in\mathbb{Z}_N^d} &\left[J(T,\frac{x}{N})\eta(T,x) - J(0,x)\eta(0,x) - \int_0^T J_t(t,\frac{x}{N})\eta(t,x)dt\right] \\ &- N^2 \sum_{x,y\in\mathbb{Z}_N^d} \int_0^T \left[e^{J(t,\frac{y}{N}) - J(t,\frac{x}{N})} - 1]\eta(t,x)(1-\eta(t,y))\pi(y-x)dt\right]\end{aligned}$$

$$E^{P_N}\big[\exp[F_J(\eta(\cdot,\cdot))]\big] = 1$$

This implies with the help of (4.3) that

$$E^{Q_N}[\frac{1}{N^d}F_J] \le \frac{1}{N^d}H(Q_N;P_N)$$

Letting $N \to \infty$ with $H = \lim_{N\to\infty} \frac{H(P_N;Q_N)}{N^d}$,

$$E^Q\Big[\int_{\mathcal{T}^d} J(T,u)\rho(T,u)du - \int_{\mathcal{T}^d} J(0,u)\rho(0,u)du - \int_0^T\int_{\mathcal{T}^d} J_t(t,u)\rho(t,u)du$$
$$- \frac{1}{2}\int_0^T\int_{\mathcal{T}^d}(\Delta_C\, J)(t,u)\rho(t,u)dtdu$$
$$- \frac{1}{2}\int_0^T\int_{\mathcal{T}^d}\langle C\nabla J, \nabla J\rangle\rho(t,u)(1-\rho(t,u))dtdu\Big] \le H$$

One of the things we should remember while carrying out large deviation estimates is that if

$$\limsup_{\lambda\to\infty}\frac{1}{\lambda}\log E^P[e^{\lambda f_i}] \le 0$$

for $i = 1, 2$, it follows that

$$\limsup_{\lambda\to\infty}\frac{1}{\lambda}\log E^P[e^{\lambda \max(f_1,f_2)}] \le 0$$

Just notice that $e^{\lambda\max(f_1,f_2)} \le e^{\lambda f_1} + e^{\lambda f_2}$ and the sum of two exponetnials that do not grow does not grow either. This is easily extended to a finite sum. It is now easy to see that if we have a family f_α of continuous functions and two sequences of probability measures P_λ and Q_λ such that $Q_\lambda \to Q$ weakly,

$$\limsup_{\lambda\to\infty}\frac{1}{\lambda}H(Q_\lambda; P_\lambda) \le H$$

and

$$\limsup_{\lambda\to\infty}\frac{1}{\lambda}\log E^{P_\lambda}[e^{\lambda f_\alpha}] \le 0$$

for every $\alpha \in A$, then of course

$$\sup_{\alpha\in A} E^Q[f_\alpha] \le H$$

But actually we get to move the sup inside for free.

$$E^Q[\sup_{\alpha\in A} f_\alpha] \le H$$

It follows from our previous discussion that the above inequality is valid if we replace A by any finite subset of A. But then by monotone convergence it is true for A. The rest is routine. If Q is any limit point of $\{Q_N\}$ with $\frac{H(Q_N;P_N)}{N^d} \to H$, we have

Lemma 4.6.5.

$$E^Q\Big[\sup_J\Big[\int_{\mathcal{T}^d} J(T,u)\rho(T,u)du - \int_{\mathcal{T}^d} J(0,u)\rho(0,u)du - \int_0^T\int_{\mathcal{T}^d} J_t(t,u)\rho(t,u)du$$
$$- \frac{1}{2}\int_0^T\int_{\mathcal{T}^d}(\Delta_C\, J)(t,u)\rho(t,u)dtdu$$
$$- \frac{1}{2}\int_0^T\int_{\mathcal{T}^d}\langle C\nabla J,\nabla J\rangle\rho(t,u)(1-\rho(t,u))dtdu\Big]\Big] \le H \tag{4.6}$$

In particular,

$$\rho_t \in L_2[[0,T];H_{-1}(\mathcal{T}^d)] \tag{4.7}$$

and

$$\rho \in L_2[[0,T];H_1(\mathcal{T}^d)] \tag{4.8}$$

Proof. We just have to make sure that estimate (4.6) is enough to provide (4.7) and (4.8). Since $\rho(1-\rho)\le 1$, (4.6) implies a bound on

$$E^Q\Big[\int_0^T \|\rho_t - \frac{1}{2}\Delta_C\,\rho\|_{-1}^2 dtdu\Big]$$
$$= E^Q\Big[\sup_J\Big[\int_{\mathcal{T}^d} J(T,u)\rho(T,u)du - \int_{\mathcal{T}^d} J(0,u)\rho(0,u)du - \int_0^T\int_{\mathcal{T}^d} J_t(t,u)\rho(t,u)du$$
$$- \frac{1}{2}\int_0^T\int_{\mathcal{T}^d}(\Delta_C\, J)(t,u)\rho(t,u)dtdu - \frac{1}{2}\int_0^T\int_{\mathcal{T}^d}\langle C\nabla J,\nabla J\rangle dtdu\Big]\Big]$$
$$\le H$$

We can do a convolution in space and time and approximate ρ by ρ^ϵ that is smooth. We will continue to have

$$E^Q\Big[\int_0^T \|\rho_t^\epsilon - \frac{1}{2}\Delta_C\,\rho^\epsilon\|_{-1}^2 dtdu\Big] \le H$$

The cross term

$$E^Q[\int_0^T\langle\rho_t^\epsilon,\Delta_C\rho^\epsilon\rangle_{-1,C}dt] = \int_0^T\int_{\mathcal{T}^d}\rho_t^\epsilon\rho^\epsilon dxdt = \frac{1}{2}\int_{\mathcal{T}^d}\rho^\epsilon(T,u)^2du - \frac{1}{2}\int_{\mathcal{T}^d}\rho^\epsilon(0,u)^2du \le \frac{1}{2}$$

and we can get a uniform bound

$$E^Q[\int_0^T\|\rho_t^\epsilon\|_{-1}^2dt + \int_0^T\int_{\mathcal{T}^d}\langle C\nabla\rho^\epsilon,\nabla\rho^\epsilon\rangle dtdu] \le H+1$$

We can now let $\epsilon\to 0$. □

4.7 Upper Bound

The proof of the upper bound is almost done. We have already shown super exponential tightness. From the exponential martingales we have (**??**). Therefore the local rate function is

$$\sup_J G_J(\rho(\cdot,\cdot))$$

where

$$\begin{aligned} G_J = &\int_{\mathcal{T}^d} \big[J(T,u)\rho(T,u) - J(0,u)\rho(0,u)\big]\,du - \int_0^T \int_{\mathcal{T}^d} J_t(t,u)\rho(t,u)dtdu \\ &- \frac{1}{2}\int_0^T \int_{\mathcal{T}^d} \Delta_C\, J(t,u)\rho(t,u)dtdu \\ &- \frac{1}{2}\int_0^T \int_{\mathcal{T}^d} \langle C\nabla J, \nabla j\rangle \rho(t,u)(1-\rho(t,u))dtdu \end{aligned}$$

The supremum is calculated as

$$\frac{1}{2}\int_0^T \|\rho_t - \frac{1}{2}\Delta_C\,\rho\|^2_{-1,C,\rho(1-\rho)}dt$$

where

$$\|f\|^2_{-1,C,\rho(1-\rho)} = \sup_J \left[2\int_{\mathcal{T}^d} J(u)f(u)du - \int_{\mathcal{T}^d} \langle C(\nabla J)(u), (\nabla J)(u)\rangle \rho(u)(1-\rho(u))du\right] \quad (4.9)$$

Finally we have to match the upper bound with the lower bound.

Lemma 4.7.1.

$$\begin{aligned} I(\rho(\cdot,\cdot)) &= \inf_{b\in\rho}\frac{1}{2}\int_0^T \int_{\mathcal{T}^d} \langle C^{-1}b(t,u), b(t,u)\rangle \rho(t,u)(1-\rho(t,u))dtdu \\ &= \frac{1}{2}\int_0^T \|\rho_t - \frac{1}{2}\Delta_C\,\rho\|^2_{-1,C,\rho(1-\rho)}dt \end{aligned}$$

Proof. First let us assume that $\rho(t,u)$ is smooth and satisfies $0 < c_1 \le \rho \le c_2 < 1$. We can do the variational problem in (4.9), $\rho_t - \frac{1}{2}\Delta_C\,\rho = f(t,u)$ being a smooth function. It results in solving

$$\nabla\cdot\rho(t,u)(1-\rho(t,u))C\nabla J = f(t,u)$$

and the rate function equals

$$\frac{1}{2}\int_0^T \int_{\mathcal{T}^d} \langle C\nabla J, \nabla J\rangle \rho(t,u)(1-\rho(t,u))dtdu$$

$b(t,u) = C\nabla J \in \mathbf{B}(\rho)$, and

$$I(\rho(\cdot,\cdot)) = \frac{1}{2}\int_0^T \int_{\mathcal{T}^d} \langle C^{-1}b, b\rangle \rho(1-\rho) dt du$$

matching the upper and lower bounds.

Finally we need to approximate an arbitrary ρ with $I(\rho(\cdot,\cdot)) < \infty$ by ρ_n that are nice such that $I(\rho_n(\cdot,\cdot)) \to I(\rho(\cdot,\cdot))$ We note that the rate function $I(\rho(\cdot,\cdot))$ is convex, lower semicontinuous and translation invariant in space and time. Smoothing by convolution will provide the needed approximation. $\square$

We have finally established the Large Deviation Principle for the processes $P_N^{\eta_N}$ on $D[[0,T];\mathcal{M}(\mathcal{T}^d)]$, under the assumption that $\frac{1}{N^d}\sum_{x\in\mathbb{Z}_N^d} \delta_{\frac{x}{N}} \eta_N(x) \to \rho_0(u)du$ weakly in $\mathcal{M}(\mathcal{T}^d)$.

Theorem 4.7.2. *The large deviation principle holds for $P_N^{\eta_N}$ with the rate function*

$$I(\rho(\cdot,\cdot)) = \frac{1}{2}\int_0^T \|\rho_t - \frac{1}{2}\Delta_C\|^2_{-1,C,\rho(1-\rho)} dt$$

provided $\rho(0,u) = \rho_0(u)$ and $+\infty$ otherwise.