Chapter 8

Existence and Uniqueness

We want to show that, under suitable conditions, given $a(s,x)$, $b(s,x)$ s_0 and x_0 there is exactly one solution $P \in \mathcal{I}(a,b)$ on $C[[s_0,T];R^d]$ such that $P[x(s_) = x_0] = 1$. We begin with a general existence theorem.

Theorem 8.1. *Let $a(s,x) = \{a_{i,j}(s,x)\}$ and $b(s,x) = \{b_j(s,x)\}$ be bounded and continuous in s and x. Then for any s_0, x_0, $\mathcal{I}(a,b)$ is nonempty.*

Proof. We follow our intuition about diffusion process and fix a time step $h = \frac{1}{n}$. We start with a Brownian motion with mean $b(s_0,x_0)$ and covariance $a(s_0,x_0)$ and run it for time $h = \frac{1}{n}$. Then conditionally we start with at time $s_0 + h$ a Brownian motion with mean $b(s_0+h, x(s_0+h))$ and covariance $a(s_0+h, x(s_0+h))$ and run it till s_0+2h. We proceed in this manner defining successive conditional distributions. This defines a probability distribution P_n on $C[[s_0,T];R^d$. If we define $a^{(n)}(s,\omega)$ and $b^{(n)}(s,\omega)$ as

$$a^{(n)}(s,\omega) = a(s_0 + jh, x(s_0 + jh, \omega) \quad \text{for } s_0 + jh \le s \le s_0 + (j+1)h$$

$$b^{(n)}(s,\omega) = b(s_0 + jh, x(s_0 + jh, \omega) \quad \text{for } s_0 + jh \le s \le s_0 + (j+1)h$$

then it is not difficult to verify that $P_n[x(s_0) = x_0] = 1$ and

$$P_n \in \mathcal{I}(a^{(n)}, b^{(n)})$$

The rest of the proof consists of verifying that $\{P_n\}$ is uniformly tight as probability measures on $C[[s_0,T];R^d]$ and any limit point P will belong to $\mathcal{I}(a,b)$. Notice that if

$$y_n(t) = x(t) - x(s_0) - \int_{s_0}^{t} b^{(n)}(s, x(s))ds$$

the difference between $y_n(\cdot)$ and $x(\cdot)$ is uniformly Lipschitz. It is therefore enough to prove the uniform tightness of $y_n(\cdot)$ under $P^{(n)}$. From the exponential martingales and Gaussian scaling it follows that

$$E^{P^{(n)}}[|y_n(t_2) - y_n(t_1)|^4] \le C|t_2 - t_1|^2$$

Now Garsia-Rodemich-Rumsey will provide tightness. Consider the functional

$$F_n(t,\omega) = f(x(t)) - f(x_0) - \int_{s_0}^{t} \frac{1}{2}\sum_{i,j} a_{i,j}(\pi_n(s), x(\pi_n(s)) f_{i,j}(x(s))ds + \int_{s_0}^{t} \sum_{j} b_j(\pi_n(s), x(\pi_n(s)) f_j(x(s))ds$$

$\{F_n\}$ are uniformly bounded and $F_n(t,\omega)$ converges uniformly on compact subsets of $C[[s_0,T];R^d]$ to

$$F(t,\omega) = f(x(t)) - f(x_0) - \int_{s_0}^{t} \frac{1}{2}\sum_{i,j} a_{i,j}(s, x(s) f_{i,j}(x(s))ds + \int_{s_0}^{t} \sum_{j} b_j(s, x(s)) f_j(x(s))ds$$

Let $s_0 \le t_1 < t_2 \le T$ and $\psi(\omega)$ a bounded continuous function on $C[[s_0,T];R^d$ that is measurable with respect to $\mathcal{F}_{t_1}$. Then if $P^{(n)} \in \mathcal{I}(a^{(n)}, b^{(n)})$

$$\int \psi(\omega) F_n(t_2,\omega) dP^{(n)} = \int \psi(\omega) F_n(t_1,\omega) dP^{(n)}$$

We can let $n \to \infty$ (along a suitable subsequence) to conclude that if P is the limit, then

$$\int \psi(\omega) F(t_2,\omega) dP = \int \psi(\omega) F(t_1,\omega) dP$$

Since $\psi(\omega)$ is an arbitrary continuous function that is measurable with respect to $\mathcal{F}_{t_1}$ this proves that $F(t,\omega)$ is a martingale for any smooth bounded f. Therefore $P \in \mathcal{I}(a,b)$. □

Let us denote by $\mathcal{I}(s_0, x_0, a, b)$, P such that $P[x(s_0) = x_0] = 1$ and $P \in \mathcal{I}(a,b)$. Let $t_1 < t_2 < t_3$. A continuous function $x(\cdot)$ on an interval $[t_1,t_3]$ can be thought of as the combination of two continuous functions, $x_1(\cdot)$ on $[t_1,t_2]$ and $x_2(\cdot)$ on $[t_2,t_3]$ with the matching condition $x_1(t_2) = x_2(t_2)$ In other words for $t < T$, the space $C[s_0,t]$ is a quotient of $C[s_0,T]$ through the natural restriction map and the fiber over $x_1(\cdot)$ is $C[t,T]$ with $x(t) = x_1(t)$. This allows us to disintegrate any probability distribution P on $C[s_0,T]$ as the marginal P_1 which is the restriction of P to $\mathcal{F}_t$ and the conditionals $\{Q_\omega\}$ on $C[t,T]$ that are continuous extensions of ω to $[t,T]$.

Theorem 8.2. *If $P \in \mathcal{I}(s_0, x_0, a, b)$ and $s_0 < t < T$, then the conditional distribution Q_ω of P given $\mathcal{F}_t$ satisfy $Q_\omega \in \mathcal{I}(t, x(t,\omega), a, b)$ for almost all ω.*

Proof. We need to prove that for almost all ω with respect to P for all $t < t_1 < t_2 \le T$ and all smooth f,

$$\int_A [F(t_2,\omega) - F(t_1,\omega)] dQ_\omega = 0$$

where $F(t,\omega)$ is as in (**??**). We can choose a countable collection of A, f, t_1, t_2 such that it is enough to verify for them. We need only check that for each choice,

$$\int_B \left[\int_A [F(t,\omega) - F(s,\omega)] dQ_\omega \right] dP = 0$$

for all $B \in \mathcal{F}_{s_0}$. But

$$\int_B \left[\int_A [F(t,\omega) - F(s,\omega)] dQ_\omega \right] dP = \int_{B \cap A} [F(t,\omega) - F(s,\omega)] dP = 0$$

□

Remark 8.1. This shows that if we have a unique solution $P_{s_0,x_0} \in \mathcal{I}(s_0, x_0, a, b)$, then $\{P_{s,x}\}$ is a Markov process with transition probability

$$p(s,x,t,A) = P_{s,x}[x(t) \in A].$$

Remark 8.2. One can repeat this argument for stopping times τ. By Doob's theorem if $Z(t)$ is a martingale then $E[Z(\tau_2) - Z(\tau_1)|\mathcal{F}_{\tau_1}] = 0$ if $s_0 \le \tau_1 \le \tau_2 \le T$. It follows that once there is uniqueness, the strong Markov property holds as well, i.e. the conditional probability

$$P|\mathcal{F}_\tau = P_{\tau,x(\tau)} \ a.e.\ P$$

on $C[\tau(\omega), T]$