# Chapter 6

# Markov Processes Kolmogorov's equations

A Markov process with values in $R^d$ can be specified by prescribing the transition probability functions $p(s,x,t,A)$ for $s < t$. They are probability measures in $A$ for fixed $s < t$ and $x$ and measurable functions of $x$ for each $s < t$ and Borel set $A$. They alsosatisfy the Chapman-Kolmogorov equations, i.e for $s < t < u$

$$p(s,x,u,A) = \int p(y,t,u,A)p(s,x,t,dy)$$

If we impose some regularity conditions on

$$\Delta_\epsilon(h) = \sup_{\substack{x \in R^d \\ 0 \le s < t \le s+h \le T}} p(s,x,t,\{y : |y-x| \ge \epsilon\}) \tag{6.1}$$

we can construct the process on $C[0,T]$ or $D[0,T]$.

**Theorem 6.1.** *If $\Delta_\epsilon(h) \to 0$ as $h \to 0$ for every $\epsilon > 0$, then for each $(s_0,x_0) \in [0,T] \times R^d$, there is a measure $P_{s_0,x_0}$ on $D[s_0,T]$ such that $P_{s_0,x_0}[x(s_0) = x_0] = 1$ and $P_{s_0,x_0}$ is a Markov Process with transition probability $p(s,x,t,y)$. In other words for $s_0 < s < t \le T$*

$$P_{s_0,x_0}[x(t) \in A|\mathcal{F}_s] = p(s,x,t,A) \ a.e.$$

*Moreover if $\Delta_\epsilon(h) = o(h)$ for every $\epsilon > 0$, as $h \to 0$, then the process $P_{s_0,x_0}$ is supported on $C[s_0,T]$.*

*Proof.* Markov property and the assumption that $\Delta_\epsilon(h) \to 0$ together imply that for every $\epsilon > 0$,

$$\lim_{h \to 0} \sup_{\substack{\omega \\ 0 \le s < t \le s+h \le T}} P[|x(s) - x(t)| \ge \epsilon|\mathcal{F}_t]$$

We saw in Theorem **??** that this is sufficient for the process to be realizable on $D[0,T]$. The condition $\Delta_\epsilon(h)=o(h)$ allows to show that

$$P[\sup_{1\le j\le \frac{T}{h}} |x(jh)-x((j-1)h)|\ge\epsilon]\le \frac{T}{h}\Delta_\epsilon(h)\to 0$$

for every $\epsilon>0$ proving that there are no jumps that are of size larger than $\epsilon$. Since $\epsilon>0$ is arbitrary the measure is supported on continuous paths. □

Let $f(x)$ be a bounded uniformly continuous function of $x$. Then

$$u(s,x)=\int f(y)p(s,x,T,dy)$$

defines a bounded function on $[0,T)\times R^d$ that satisfies $\sup_x |u(s,x)-f(x)|\to 0$ as $s\to T$. For any $\epsilon>0$,

$$\begin{aligned}\sup_x |u(s,x)-f(x)| &\le 2\sup_x |f(x)|\sup_x p(s,x,T,\{y:|y-x|\ge\epsilon\})\\ &\quad+\sup_{|x-y|\le\epsilon}|f(x)-f(y)|\\ &\le 2\sup_x|f(x)|\Delta_\epsilon(T-s)+\sup_{|x-y|\le\epsilon}|f(x)-f(y)|\end{aligned}$$

We can first let $s\to T$ and then $\epsilon\to 0$. If we want to consider a diffusion process with its increments $x(s+h)-x(s)$ having a conditional distribution that is Gaussian with expectation $\{hb_j(s,x)\}$ and covariance $\{ha_{i,j}(s,x)\}$, it is natural to demand that for $1\le j\le d$

$$\lim_{h\to 0}\frac{1}{h}\int_{|y-x|\le\ell}\int (y_j-x_j)p(s,x,s+h,dy)=b_j(s,x)$$

and for $1\le i,j\le d$,

$$\lim_{h\to 0}\frac{1}{h}\int_{|y-x|\le\ell}\int (y_i-x_i)(y_j-x_j)p(s,x,s+h,dy)=a_{i,j}(s,x)$$

and the condition $\Delta_\epsilon(h)=o(h)$ guarantees that if the limits exist for some $\ell\in(0,\infty)$, then they do for every $\ell>0$.

**Theorem 6.2.** *If $u(s,x)$ has one bounded continuous $s$ derivative and two bounded continuous $x$ derivatives, then it satisfies the Kolmogorov backward equation*

$$\frac{\partial u}{\partial s}+\sum_j b_j(s,x)\frac{\partial u}{\partial x_j}(s,x)+\frac{1}{2}\sum_{i,j}a_{i,j}(s,x)\frac{\partial^2 u}{\partial x_i\partial x_j}(s,x)=0$$

*with the boundary condition $u(s,x)\to f(x)$ as $s\to T$.*

*Proof.* We use the Chapman-Kolmogorov equation to express

$$u(s,x) = \int u(s+h,y)p(s,x,s+h,dy)$$

and the Taylor expansion to write

$$\begin{aligned} u(s+h,y) = u(s,x) + h\,u_s(s,x) + \sum_j u_{x_j}(s,x)(y_j - x_j) \\ + \frac{1}{2}\sum_{i,j} a_{i,j}(s,x)u_{x_i,x_j}(s,x)(y_i - x_i)(y_j - x_j) + Error \end{aligned}$$

If $|y-x| \le \ell$, the the error is bounded by $\delta(\ell)|y-x|^2$ where $\delta(\ell) \to 0$ with $\ell$. We can split the integral over $|x-y| \le \ell$ and $|x-y| > \ell$, divide by $h$ and let $h \to 0$. Because $\Delta_\ell(h) = o(h)$, the contribution from $|x-y| \ge \ell$ goes to 0 even when divided by $h$. The first three terms yield

$$\frac{\partial u}{\partial s} + \sum_j b_j(s,x)\frac{\partial u}{\partial x_j}(s,x) + \frac{1}{2}\sum_{i,j} a_{i,j}(s,x)\frac{\partial^2 u}{\partial x_i \partial x_j}(s,x)$$

and the error term can contribute at most $C\delta(\ell)\sup_{s,x}|a_{i,j}(s,x)|$. Since $\ell$ can be arbitrarily small and $\delta(\ell) \to 0$ as $\ell \to 0$ we are done. □

If the Markov Process has stationary transition probabilities, i.e $p(s,x,t,A) = p(t-s,x,A)$ depends only on the difference $t-s$, then $b_j(s,x)$ and $a_{i,j}(s,x)$ are independent of $s$ and the backward equation takes the form

$$u_s + \sum_j b_j(x)\frac{\partial u}{\partial x_j}(s,x) + \frac{1}{2}\sum_{i,j} a_{i,j}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(s,x)$$

It is now possible to write $u(T-t,x)$ instead of $u(t,x)$ and the equation becomes

$$u_t = \sum_j b_j(x)\frac{\partial u}{\partial x_j}(s,x) + \frac{1}{2}\sum_{i,j} a_{i,j}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(s,x) = \mathcal{L}u$$

and $u(t,x) \to f(x)$ as $t \to 0$. $u(t,x)$ is of course now given by

$$u(t,x) = \int f(y)p(t,x,dy)$$

The basic question in the study of diffusion processes is one of associating in a some canonical fashion a Markov process for a given set of coefficients. This could be done in many ways. Given $\{b_j(s,x)\}, \{a_{i,j}(s,x)\}$ one can ask for a family $P_{s,x}$ of probability measures on $C[[s,T];R^d]$ such that $P_{s,x}[x(s) = x] = 1$ and $P_{s,x} \in \mathcal{I}(a(t,x(t)), b(t,x(t)))$ on $C[[s,T],\mathcal{F}_t]$. We would expect $\{P_{s,x}\}$ to be a Markov family with transition probability

$$p(s,x,t,A) = P_{s,x}[x(t) \in A]$$

One way of constructing $p(s,x,t,A)$ is to solve Kolmogorov's equation

$$u_s + \sum_j b_j(s,x)\frac{\partial u}{\partial x_j}(s,x) + \frac{1}{2}\sum_{i,j} a_{i,j}(s,x)\frac{\partial^2 u}{\partial x_i \partial x_j}(s,x) = 0$$

for $s < t$ with boundary condition $u(t,x) = f(x)$. the solution $u(s,x)$ will be unique under suitable assumptions, and the map $f \to u$ is a linear map. Moreover the maximum principle for parabolic equations will guarantee that the linear functional $u(t,x) = \Lambda_{s,x,t}(f)$ is nonnegative and we can use Riesz representation theorem to express

$$u(s,x) = \int f(y)p(s,x,t,dy)$$

In PDE theory one obtains, under suitable conditions, a "fundamental solution" $p(s,x,t,y)$ that satisfies for $s < t, y \in R^d$

$$p_s(s,x) + \sum_j b_j(s,x)\frac{\partial p}{\partial x_j}(s,x) + \frac{1}{2}\sum_{i,j} a_{i,j}(s,x)\frac{\partial^2 p}{\partial x_i \partial x_j}(s,x) = 0$$

and $p(s,x,t,\cdot) \to \delta_x(\cdot)$ as $s \to t$. $p(s,x,t,y)dy$ will satisfy the Chapman-Kolmogorov equations and can be used to construct a Markov process with continuous trajectories. This will define the Diffusion process corresponding to $[\{a_{i,j}(t,x)\},\{b_j(t,x)\}]$. The conditions on the coefficients for this approach to work are:

- There exist constants $C < \infty$ and $0 < \alpha \le 1$ such that

  $$\sup_{i,j}|a_{i,j}(s,x) - a_{i,j}(t,y)| + \sup_j |b_j(s,x) - b_j(t,y) \le C[|t-s|^\alpha + |x-y|^\alpha]$$

- For som constant $C < \infty$

  $$\sup_{i,j,s,x}|a_{i,j}(s,x)| + \sup_{j,s,x}|b_j(s,x)| \le C$$

- The symmetric matrices $\{a_{i,j}(s,x)\}$ are all uniformly elliptic, i.e. uniformly positive definite. For some constant $c > 0$,

  $$\sum_{i,j} a_{i,j}(s,x)\xi_i\xi_j \ge c\sum_j \xi_j^2$$

  for all $s,x$ and $\{\xi_j\} \in R^d$.

A proof can be found in PDE texts, for example in Avner Friedman's book "Partial Differential Equations of Parabolic Type". Once we have the fundamental solution, the backward equation is easily solved, by the formula

$$u(s,x) = \int f(y)p(s,x,T,y)dy$$

Another approach due to Itô to construct a diffusion process is to start with a Brownian motion $(\Omega, \mathcal{F}_t, \beta(t), P)$ and try to solve the equation

$$dx(t) = \sigma(t, x(t)) \cdot d\beta(t) + b(t, x(t))dt$$

where $\sigma(s,x)\sigma^*(s,x) = a(s,x)$. By a solution we mean an almost surely continuous process $x(t,\omega)$ that is progressively measurable on $(\Omega, \mathcal{F}_t, P)$ that satisfies for $t \geq s_0$

$$x(t,\omega) = x(s_0,\omega) + \int_{s_0}^t \sigma(s,\omega) \cdot d\beta(s) + \int_0^t b(s, x(s))ds \tag{6.2}$$

Under the following hypotheses

- There exist constant $C$ such that

$$\sup_{i,j,s} |\sigma_{i,j}(s,x) - \sigma_{i,j}(s,y)| + \sup_{j,s} |b_j(s,x) - b_j(s,y)| \leq C|x-y|$$

- There exist constant $C$ such that

$$\sup_{i,j,s,x} |\sigma_{i,j}(s,x)| + \sup_{j,s,x} |b_j(s,x)| \leq C$$

according to Itô's theory, for every $s_0$ and $\xi(\omega) : \Omega \to\in R^d$ that is measurable with respect to $\mathcal{F}_{s_0}$ and satisfies $E[\|\xi\|^2] < \infty$ there is a unique solution of (6.2), with $x(s_0,\omega) = \xi(\omega)$. In particular if we solve with $\xi(\omega) = x$ with probability 1, then we obtain the transition probbaility

$$p(s_0, x, t, A) = P[x(t) \in A | x(s_0) = x]$$

There is also the approach of trying to solve the Kolmogorov equation in some generalized sense, i.e in a suitable Sobolev space. Of course if more than one method work they should lead to the same answer! We will use the following formulation. We are given the functions $\{a_{i,j}(s,x)\}, \{b_j(s,x)\}$ that are uniformly bounded and measurable. For given $(s_0, x_0)$ we can look for a probability measure $P = P_{s_0,x_0}$ on $C[[s_0,T]; R^d]$ such that $P_{s_0,x_0}[x(s_0) = x_0] = 1$ and $x(s) \in \mathcal{I}(\{a_{i,j}(s, x(s))\}, \{b_j(s, x(s))\})$. We can use the as definition

$$f(t, x(t)) - f(s_0, x_0) - \int_{s_0}^t [f_s + (L_s f)(s, x(s))]ds$$

is a martingale with respect to $(\Omega, \mathcal{F}_t, P)$ for $t \geq s_0$, where

$$(L_s f)(s,x) = \frac{1}{2} \sum_{i,j} a_{i,j}(s,x) f_{i,j}(s,x) + \sum_j b_j(s,x) f_j(s,x)$$

Under suitable conditions we would like to prove that $P$ exists, is unique and is a Markov process with transition probabilities given by

$$p(s_0, x_0, t, A) = P_{s_0,x_0}[x(t) \in A]$$

It will turn out that if any other method works then the formulation in terms of martingales will also work.