# Chapter 17

# Reflection in higher dimensions

In one dimension there is only one direction to come back from the boundary. In higher dimensions one can come in at angle which is not necessarily the normal. If we consider for simplicity Brownian motion is the half space $x \geq 0$ in the plane, when the process is at the boundary it can be reflected in the normal direction $(1, 0)$ or at an angle, still pointing into the region $(1, a)$. In terms of PDE, the boundary condition at $x = 0$ will be $u_x + au_y = 0$. If $a = 0$ then the two components are independent and the process is $(x(t), y(t))$ where $x(t)$ is the reflected one dimensional Brownian Motion and $y(t)$ is an ordinary Brownian motion independent of it. $a \neq 0$, then $y(t)$ gets a push when $x(t)$ at 0. The push is $aA(t)$ where $A(t)$ is the local time at 0. So we have explicit representation of the process as

$$x(t) = \beta_1(t) + A(t)$$
$$y(t) = \beta_2(t) + aA(t)$$

We can even let $a$ depend on $y$ so that the boundary condition is $u_x+a(y)u_y = 0$. So long as $a(y)$ is bounded the direction always points inside the domain. One can try to solve the equation

$$dy(t) = \beta_2(t) + \int_0^t a(y(s))dA(s)$$

If $a(y)$ is Lipschitz one can solve this by iteration and get a unique solution. Even if the process in the interior is Brownian motion with a covariance different from $I$, i.e. the components are dependent one can always make a linear transformation and reduce the problem to the independent case. If we have variable coefficient, for instance continuous coefficients, then perturbation techniques in PDE can be used to show that the process exists and is unique, if the boundary is smooth and the boundary condition is of the form $\langle J, \nabla u\rangle = 0$, where $J$ is normalized so that $\langle J, n\rangle = 1$, $n$ being the inward normal. There is some trade off between smoothness of $a$ and the smoothness of $J$. The reflected processes in the region $G$ with boundary $\delta G$ can be characterized by the condition

$$f(x(t)) - f(x(0)) - \int_0^t (\mathcal{L}f)(x(s))ds$$

is a sub-martingale provided $\langle J, \nabla u\rangle \geq 0$ on $\delta G$. $\mathcal{L}$ is the generator

$$\frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(x)\frac{\partial}{\partial x_i \partial x_j} + \sum_{j=1}^{d} b_j(x)\frac{\partial}{\partial x_j}$$

in the interior $G$. One can replace the normal $n$ by $\nabla\phi$ where $\phi$ is a smooth function, with $\phi > 0$ in $G$, $\phi = 0$ on $G$ and $\|\nabla\phi\| > 0$ on $\delta G$. There are corresponding versions that are slowed down on the boundary and spend positive time there, as well as versions that are explicitly time dependent.

**An example.**

We will illustrate the ideas by looking at an example coming from queuing theory. Consider two arrival streams for two servers that form two queues. The Poisson arrival rates for them are $\lambda_1(N)$ and $\lambda_2(N)$ respectively, where $N$ is a large parameter signifying heavy traffic. The corresponding service times are $(\mu_1(N))^{-1}$ and $(\mu_2(N))^{-1}$. The system is operating near capacity so that $\lambda_i(N) = \lambda_i N^2$ and $\mu_i(N) = \lambda_i N^2 + \alpha_i N$. Behavior of the system will depend on the protocol. Let us suppose that when one queue is empty that server will help out the server of the second queue and the service rate for the nonempty queue is now double the normal rate. We have a Markov chain in continuous time. State space is $Z^+ \times Z^+$, i.e the two queue lengths $i, j \geq 0$. The transitions are

$$\begin{aligned}(i,j) &\to (i+1,j) \quad \text{with rate } \lambda_1(N)\\ (i,j) &\to (i,j+1) \quad \text{with rate } \lambda_2(N)\end{aligned}$$

for all $i, j$.

$$\begin{aligned}(i,j) &\to (i-1,j) \quad \text{with rate } \mu_1(N)\\ (i,j) &\to (i,j-1) \quad \text{with rate } \mu_2(N)\end{aligned}$$

if $i, j > 0$. If either $i$ or $j$ equlas 0

$$\begin{aligned}(i,0) &\to (i-1,0) \quad \text{with rate } 2\mu_1(N)\\ (0,j) &\to (0,j-1) \quad \text{with rate } 2\mu_2(N)\end{aligned}$$

If we measure the queue lengths as $x_1 = \frac{i}{N}$ and $x_2 = \frac{j}{N}$, the generator at $x_1 > 0, x_2 > 0$ is

$$
\begin{aligned}
(\mathcal{A}_N f)(x_1,x_2) = \lambda_1(N)[f(x_1+\frac{1}{N},x_2) - f(x_1,x_2)]\\
+\lambda_2(N)[f(x_1,x_2+\frac{1}{N}) - f(x_1,x_2)]\\
+\mu_1(N)[f(x_1-\frac{1}{N},x_2) - f(x_1,x_2)]\\
+\mu_2(N)[f(x_1,x_2) - f(x_1,x_2-\frac{1}{N})]
\end{aligned}
$$

On the other hand if $x_1 = 0, x_2 > 0$

$$
\begin{aligned}
(\mathcal{A}_N f)(0,x_2) = \lambda_1(N)[f(\frac{1}{N},x_2) - f(0,x_2)] + \lambda_2(N)[f(0,x_2+\frac{1}{N}) - f(0,x_2)]\\
+2\mu_2(N)[f(0,x_2) - f(x_1,x_2-\frac{1}{N})]
\end{aligned}
$$

and a similar expression if $x_1 > 0, x_2 = 0$. As $N \to \infty$, if $x_1, x_2 > 0$,

$$(\mathcal{A}_N f)(x_1,x_2) \to -\alpha_1 f_1 - \alpha_2 f_2 + \lambda_1 f_{11} + \lambda_2 f_{22}$$

If $x_1 = 0$

$$\frac{1}{N}(\mathcal{A}_N f)(0,x_2) \to \lambda_1 f_1(0,x_2) - \lambda_2 f_2(0,x_2)$$

and a similar condition when $x_2 = 0$. It is therefore worth considering the following problem in the domain $G = \{(x_1,x_2) : x_1 \geq 0, x_2 \geq 0\}$. The operator in the interior is

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{2} a_{i,j}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{2} b_j \frac{\partial}{\partial x_j}$$

The boundary conditions are $\frac{\partial}{\partial x_1} + \gamma_1 \frac{\partial}{\partial x_2} = 0$ on $x_1 = 0$ and $\frac{\partial}{\partial x_2} + \gamma_2 \frac{\partial}{\partial x_1} = 0$ on $x_2 = 0$. We can do a linear change of variables and reduce the generator in the interior to be $\Delta$. But the domain will change from a quadrant to a wedge $0 \leq \theta \leq \alpha$. Locally we have no problem unless the path gets to the origin $(0,0)$ which is a singular point on the boundary. Whether the point is reached or not will only depend on the angle $\alpha$ of the wedge and the two directions of reflection along the two sides of the wedge. Girsanov's theorem shows that $b_1, b_2$ will not affect the qualitative behavior. Let us take them to be 0. We need to determine when the origin will be reached with positive probability. Consider a homogeneous function of order $\alpha$.

$$H(x_1,x_2) = r^\kappa h(\theta)$$

defined on the first quadrant. We want this to be a solution of

$$\Delta H = 0,$$

with the boundary conditions being satisfied. This will lead to an ODE for $h(\theta)$ on $[0,\alpha]$ with boundary conditions at the ends. $\kappa$ is determined by the condition that $h$ be positive. If $\kappa > 0$, origin will be reached and if $\kappa < 0$ it will not be. $\kappa = 0$ corresponds to a solution with a logarithmic singularity and again $(0,0)$ will not be reached. The equation

$$\Delta r^\kappa h(\theta) = h(\theta)\Delta r^\kappa + r^\kappa \Delta h(\theta) + 2\nabla r^\kappa \cdot \nabla h(\theta) = 0$$

reduces to the following ODE for $h(\theta)$ on $0 \le \theta \le \alpha$.

$$k^2 h(\theta) + h''(\theta) = 0 \tag{17.1}$$

with boundary conditions specified by the two angles $\alpha_1$ and $\alpha_2$, that can vary between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, that the directions of reflection make with the respective inward normals. Derivatives in the direction $\frac{\pi}{2} + \alpha_1$ along $\theta = 0$ and in the direction $\alpha - \frac{\pi}{2} + \alpha_2$ along $\theta = \alpha$ must be equal to 0. There will be exactly one value of $k$ for which we will have a solution that is positive on $[0, \alpha]$. Accessibility will depend on the sign of $k$. One can show that, even when accessible, there is only one solution if we insist that the total amount of time the process spends at 0, i.e. the Lebesgue measure of $\{t : x(t) = 0\}$ is almost surely 0.

**Test for accessibility of 0.** We need to compute the gradients in the specified directions along the corresponding rays in polar coordinates. Gradient in the direction $\phi$ at the point $(x, y)$ is

$$\cos\phi \frac{\partial}{\partial x} + \sin\phi \frac{\partial}{\partial y} = \cos\phi[\frac{x}{r}\frac{\partial}{\partial r} - \frac{y}{r^2}\frac{\partial}{\partial \theta}] + \sin\phi[\frac{y}{r}\frac{\partial}{\partial r} + \frac{x}{r^2}\frac{\partial}{\partial \theta}]$$

Along the line $\theta = 0$ it reduces to $\cos\phi \frac{\partial}{\partial r} + \frac{\sin\phi}{r}\frac{\partial}{\partial \theta}$ whereas along the line $\theta = \alpha$ we end up with $\cos(\phi - \alpha)\frac{\partial}{\partial r} + \frac{\sin(\phi-\alpha)}{r}\frac{\partial}{\partial \theta}$. When applied to the function $r^k[A\cos k\theta + B\sin k\theta]$, which is the general solution of (**??**) the boundary condition at $\theta = 0$ (with $\phi = \frac{\pi}{2} + \alpha_1$), for the function $h(\theta) = A\cos k\theta + B\sin k\theta$ leads to

$$-A\sin\alpha_1 + B\cos\alpha_1 = 0$$

Similarly the boundary condition at $\theta = \alpha$ (with $\phi = \alpha - \frac{\pi}{2} + \alpha_2$) is

$$-A\sin(k\alpha + \alpha_2) + B\cos(k\alpha + \alpha_2) = 0$$

For these two equations to have a nontrivial solution we need

$$\cos\alpha_1 \sin(k\alpha + \alpha_2) - \sin\alpha_1 \cos(k\alpha + \alpha_2) = \sin(k\alpha + \alpha_2 - \alpha_1) = 0$$

For some ineteger $r$,

$$k = \frac{\alpha_1 - \alpha_2 + r\pi}{\alpha}$$

The solution is $h(\theta) = \cos(k\theta - \alpha_1)$. Have to pick $r$ so that $h$ does not vanish in $[0, \alpha]$. Only $r = 0$ will work, forcing $k = \frac{\alpha_1 - \alpha_2}{\alpha}$.