Chapter 14

The General Case

First some notation. Given coefficients $[a, b]$, we denote by $\mathcal{L}_s^{a,b}$ the operator

$$(\mathcal{L}_s^{a,b} f)(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(s,x) f_{i,j}(x) + \sum_{j=1}^{d} b_j(s,x) f_j(x)$$

We denote by $\mathcal{C}(a, b, s_0, x_0)$ the space of all solutions to the martingale problem corresponding to these coefficients that start from (s_0, x_0), i.e the space of all stochastic processes that satisfy $P[x(s_0) = x_0] = 1$ and

$$f(x(t)) - f(x(s_0)) - \int_{s_0}^{t} \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(s, x(s)) f_{i,j}(x(s)) ds$$

is a martingale with respect $(\Omega, \mathcal{F}_t^{s_0}, P)$.

Our goal is to prove that if $a_{i,j}(t, x)$ is continuous, uniformly bounded and strictly elliptic, i.e. nonsingular for every (t, x) and $b_j(t, x)$ are bounded and measurable, then for every (s_0, x_0) $\mathcal{C}(a, b, s_0, x_0)$ consists of a unique element $P = P_{s_0,x_0}$ and the family $\{P_s, x\}$ is a strong Markov family.

Lemma 14.1 (Principle of Localization)**.** *We have an operator*

$$(\mathcal{L}_s f)(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(s,x) f_{i,j}(x) + \sum_{j=1}^{d} b_j(s,x) f_j(x)$$

with coefficients that are locally bounded. We want to show that for any (s_0, x_0) there is at most one solution starting from (s_0, x_0), i.e. $\mathcal{C}(a, b, s_0, x_0)$ consists of at most one element. Suppose we have a collection $\{a_{s,x}(t, y), b_{s,x}(t, y)\}$ of coefficients with the following properties:

- *For each (s, x) there is $\epsilon(s, x) \geq \epsilon(K) > 0$ which is uniformly positive on compact sets such that $a(t, y) = a_{s,x}(t, y), b_{s,x}(t, y) = b(t, y)$ provided*

$|s-t|+|x-y| \le \epsilon(s,x)$, i.e $[a,b]$ and $[a_{s,x},b_{s,x}]$ agree in an $\epsilon(s,x)$ neighborhood of (s,x). In other words we can think of $[a_{s,x},b_{s,x}]$ as a modification of $[a,b]$ outside an $\epsilon(s,x)$ neighborhood of (s,x).

- *For every (s,x), $C(a_{s,x},b_{s,x},s_0,x_0)$ consists of exactly one solution $\{P^{s,x}_{s_0,x_0}\}$*

Then for every (s_0,x_0), $\mathcal{C}(a,b,s_0,x_0)$ consists of at most one element. Local uniqueness implies global uniqueness.

Proof. We proceed in steps.

Step 1. Let (s_0,x_0) be arbitrary and let $P_1,P_2 \in \mathcal{C}(a,b,s_0,x_0)$. Then if $\tau = \inf\{t \ge s_0 : |x(t)-x_0|+|t-s| \ge \epsilon(s_0,x_0)\}$, then we will show that $P_1 = P_2$ on $\mathcal{F}^{s_0}_\tau$. To accomplish thsis we define new processes Q_1,Q_2 by taking $Q_i = P_i$ on $\mathcal{F}^{s_0}_\tau$ and $Q_1\big|\mathcal{F}^\tau_\infty = Q_2\big|\mathcal{F}^\tau_\infty = P^{s_0,x_0}_{\tau,x(\tau)}$. In other words after time τ we replace P_1 and P_2 by the solution for $[a_{s_0,x_0},b_{s_0,x_0}]$. Since $[a,b]=[a_{s_0,x_0},b_{s_0,x_0}]$ until the exit time from the ball of radius $\epsilon(s_0,x_0)$ around (s_0,x_0) both Q_1 and Q_2 are in $\mathcal{C}(a_{s_0,x_0},b_{s_0,x_0},s_0,x_0)$ which has exactly one element. Therefore $Q_1 = Q_2$ which implies that $P_1 = P_2$ on $\mathcal{F}^{s_0}_\tau$.

We have (indirectly) used the following (elementary) fact. Let Q be a probability measure on $(\Omega,\mathcal{F}_t)$ and $Z(t,\omega)$ progressively measurable function. Let τ be a stopping time and Q^τ_ω be the regular conditional probability distribution of $Q\big|\mathcal{F}_\tau$. Suppose $Z(t)-Z(\tau)$ is a martingale for $t \ge \tau(\omega)$ with respect to Q^τ_ω for almost all ω and $Z(\tau \wedge t)$ is martingale with respect to Q, then $Z(t)$ is a martingale with respect to Q. This is needed to provide a formal proof that $Q_1,Q_2 \in \mathcal{C}(a_{s_0,x_0},b_{s_0,x_0},s_0,x_0)$.

Step 2. Define successively $\tau_0 = s_0$ and for $n \ge 1$,

$$\tau_n = \inf\{t : t \ge \tau_{n-1}, |t-\tau_{n-1}| + |x(t)-x(\tau_{n-1})| \ge \epsilon(t_{n-1},x(t_{n-1})\}$$

By induction we can show that if $P_1,P_2 \in C(a_{s,x},b_{s,x},s_0,x_0)$, then $P_1 = P_2$ on $\mathcal{F}^{s_0}_{\tau_n}$. The induction step assumes that this is true for $\mathcal{F}^{s_0}_{\tau_{j-1}}$. For almost all ω with respect to P_1,P_2, the conditionals $Q^{1,\tau_{j-1}}_\omega, Q^{2,\tau_{j-1}}_\omega$ are both members of $\mathcal{C}(a_{\tau_{j-1},x(\tau_{j-1})},b_{\tau_{j-1},x(\tau_{j-1})},\tau_{j-1},x(\tau_{j-1}))$. Therefore they agree on $\mathcal{F}^{\tau_{j-1}(\omega)}_{\tau_j}$ for almost all ω. Thus $P_1 = P_2$ on $\mathcal{F}_{\tau_j}$ and the induction works.

Step 3. We show that $\tau_n \to \infty$ a.e. with respect to both P_1 and P_2. From the continuity of paths it is clear that if τ_n tends to a finite limit, then $\epsilon(\tau_n,x(\tau_n))$ must go to 0 and this happens only when $x(\tau_n) \to \infty$. Since the trajectories are continuous this forces $\tau_n \to \infty$. If $A \in \mathcal{F}^{s_0}_t$ then $A \cap \{\tau_n \ge t\} \in \mathcal{F}^{s_0}_{\tau_n}$ and $P_1 = P_2$ on that set. Therefore

$$|P_1(A)-P_2(A)| \le P_1(A\cap\{\tau_n \le t\} + P_2(A\cap\{\tau_n \le t\} \le P_1[\tau_n \le t] + P_2[\tau_n \le t] \to 0$$

as $n \to \infty$. But the left hand side is independent of n and so must equal 0.

□

Lemma 14.2. *For each positive definite symmetric matrix $\bar{a}$ there is an $\epsilon(\bar{a})$ that depends only on the dimension d and lowest and highest eigenvalues of $\bar{a}$, such that if $\|a_{i,j}(t,x)-\bar{a}_{i,j}\| \le \epsilon(\bar{a})$, then for any (s_0,x_0), $\mathcal{C}(a,0,s_0,x_0)$ consists of a single measure $P^a_{s_0,x_0}$. The family $\{P^a_{s_0,x_0}\}$ depends continuously on (s_0,x_0)*
.

Proof. The proof depends on the following estimate. Consider the fundamental solution of the heat equation

$$p(s,x,t,y) = (2\pi(t-s))^{-\frac{d}{2}} \exp[-\frac{\|y-x\|^2}{2(t-s)}]$$

and the associated Greens operator

$$(Gf)(s,x) = \int_s^\infty \int_{R^d} f(t,y)p(s,x,t,y)dtdy$$

Then if f is supported on $[0,T]\times R^d$, we have with $\frac{1}{k}+\frac{1}{k^*}=1$ and $d(k-1)<2$ i.e., $k^* > \frac{d+2}{2}$

$$\begin{aligned}|(Gf)(s,x)| &\le \left[\int_s^T \int_{R^d} |p(s,x,t,y)|^\kappa dtdy\right]^{\frac{1}{\kappa}} \left[\int_s^T \int_{R^d} |f(t,y)|^{\kappa^*} dtdy\right]^{\frac{1}{\kappa^*}} \\ &\le C_{d,\kappa}(T-s)\|f\|_{\kappa^*}\end{aligned}$$

Moreover according to a theorem of B.F. Jones if $u(s,x)=(Gf)(s,x)$, then for any $p\in(1,\infty)$, there is a constant C_p such that for any T,

$$\|u_t\|_p + \sum_{i,j}\|u_{i,j}\|_p \le C_p\|f\|_p$$

In particular there is a $\delta>0$ such that if $\sup_{i,j}\|a_{i,j}(s,x)-\delta_{i,j}\| \le \delta$, then the equation

$$u_s(s,x) + \frac{1}{2}\sum_{i,j} a_{i,j}(s,x)u_{i,j}(s,x) = f(s,x) \quad \text{on } [0,T]\times R^d; \quad u(T,x)\equiv 0$$

can be solved by perturbation as

$$u = G(I + \frac{1}{2}[a_{i,j}(\cdot,\cdot)-\delta_{i,j}](Gf)_{i,j})^{-1} f$$

For any d we need to have $p>k^*$ and δ small enough for the perturbation to work for such a p. Then the solution u will be in the Sobolev space $W^p_{1,2}$ and $\sup_{\substack{0\le t\le T\\ x\in R^d}} |u(t,x)| \le c(T)\|f\|_p$.

The rest of the proof proceeds exactly like the one dimensional case, or the stationary two dimensional case. The only difference now is that we need $\sup_{i,j}\|a_{i,j}(s,x)-\delta_{i,j}\| \le \delta$. By a linear transformation we can replace $\delta_{i,j}$ by any constant coefficients $\bar{a}_{i,j}$ so long as it is elliptic. δ will depend on the ellipticity and will be uniformly positive so long as $\bar{a}$ remains uniformly elliptic, i.e. the eigenvalues have a uniform upper bound and uniform positive lower bound. □

We are now ready to state and prove the main theorem.

Theorem 14.3. *Let $\{a_{i,j}(s,x)\}$ be continuous, positive definite for each (s,x) and satisfy the growth condition $|a_{i,j}(s,x)| \leq C(1+|x|^2)$, while $\{b_i(s,x)\}$ are measurable and satisfy the growth condition $|b_i(s,x)| \leq C(1+|x|)$. Then for every (s_0,x_0) there is a unique element P_{s_0,x_0} in $\mathcal{C}(a,b.s_0,x_0)$ which will be a Markov process with transition probability*

$$p(s,x,t,A) = P_{s,x}[x(t) \in A]$$

Proof. First we show that we can assume with out loss of generality that a and b are bounded. Otherwise we can modify them outside $\{x : |x| \leq \ell\}$ so that the modified coefficients $[a^\ell, b^\ell]$ have a unique solution $P^\ell \in \mathcal{C}(a^\ell, b^\ell, s_0, x_0)$. Any solution in $P \in \mathcal{C}(a,b,s_0,x_0)$ must agree with P^ℓ on $\mathcal{F}^{s_0}_{\tau_\ell}$ where τ_ℓ is the exit time from the ball of radius ℓ. For any solution P, by continuity of paths $\tau_\ell \to \infty$ a.e. and therefore P is unique if it exists. This is true with out any growth conditions. We need to prove existence under growth conditions. This needs an estimate

$$\lim_{\ell\to\infty} P^\ell[\tau_\ell \leq t] = 0$$

Such an estimate would imply that for $A \in \mathcal{F}^{s_0}_t$,

$$\lim_{\ell_1,\ell_2\to\infty} |P^{\ell_1}(A) - P^{\ell_2}(A)| \leq \lim_{\ell_1,\ell_2\to\infty} 2P^{\ell_1\wedge\ell_2}[\tau_{\ell_1\wedge\ell_2} \leq t] = 0$$

proving the existence of a limit P of P^ℓ which can be easily verified to be in $\mathcal{C}(a,b,s_0,x_0)$ To this end we consider the function $u(x) = (1+|x|^2)$. From the bounds on a, b

$$u_s + \frac{1}{2}\mathcal{L}_s u \leq C(1+|x|^2) \leq Cu$$

By Itô's formula, if τ_ℓ is the exit time from the ball of radius ℓ,

$$E[u((x(\tau))e^{-C\tau_\ell}] \leq u(x_0)$$

Therefore

$$E^P[e^{-C\tau_\ell}] \leq u(x_0)(1+\ell^2)^{-1} \to 0$$

as $\ell \to \infty$ implying $P[\tau_\ell \leq t] \to 0$ as $\ell \to \infty$ for any fixed t. It remains to prove uniqueness for a, b bounded. For proving uniqueness there is no loss of generality in assuming a is uniformly elliptic, because we can modify it ourside a ball of radius ℓ. If it is uniformly elliptic, by Girsanov's formula we can assume $b = 0$. Now lemmas 14.1 and 14.2 complete the proof. ∎