1. Application.

We now present an application of the methods of large deviations. Let us consider the usual random walk on Z^d. Suppose certain sites in Z^d were traps. The sites that are traps are chosen randomly and the probability of a given site being a trap is p and for different sites the choices are made independently. The "environment " is random Bernoulli where the proportion p of sites that are traps is some fixed $0 < p < 1$. Although we could look at random walk in discrete time, it is slightly more convenient to study the continuous time situation where the jumps to the nearest neighbor sites are made after successive independent exponential waiting times with mean $\frac{1}{d}$.

The continuous time random walk in d-dimensions has generator

$$\mathcal{L}f(x) = \frac{1}{2} \sum_{y:y\sim x} [f(y) - f(x)]$$

where $[y : y \sim x]$ refers to the $2d$ neighbors of x. The random walk is terminated as soon as a trap is encountered. The penalty is rather severe. We want to estimate the probability

$$\Pr \Big[\text{ The random walk avoids a trap during } [0,t] \Big]$$

especially its decay rate as $t \to \infty$. Since it perfectly safe for the process to revisit any site that it has already visited and survived, we can calculate the above probability as $E\big[p^{\xi(t)}\big]$ where $\xi(t)$ is the number of distinct sites visited by the random walk. The problem then clearly is to estimate this expectation. To get a lower bound on the probability is to make sure that the number of distinct sites visited is rather small or at least not too large. Trying to limit the number of steps is one way. Maybe the random walk did not take too many steps. This is rather unlikely because the number of steps is Poisson with parameter t and the probability that it takes value smaller than $(1 - \delta)t$ is exponentially small in t for any $\delta > 0$. On the other hand it might better to confine it to a small region around the starting point. To confine it to a ball of radius $\sqrt{t}$ is no big deal because it is the normal behavior. However confining it to a ball of radius t^α for some $\alpha < \frac{1}{2}$ is going be an event of small probability. To calculate this probability, we will see by LDP that the probability of confinement is roughly $\exp[-\lambda t]$ where λ is the smallest Dirichlet eigen value for the operator $-\mathcal{L}$ for the region in question. For a ball of radius rt^α the operator $\mathcal{L}$ should be well appproximated by Δ the Laplacian and $\lambda \equiv c_d(rt)^{-2\alpha}$ for some $c_d > 0$. The number of lattice sites in a ball of radius $(rt)^\alpha$ is of course $v_d(rt)^{\alpha d}$ where v_d is the volume of the unit ball in d dimensions. We clearly have then a lower bound

$$E\big[p^{\xi(t)}\big] \geq \exp[v_d(\log p)(rt)^{\alpha d} - c_d r^{-2} t^{1-2\alpha}]$$

It is easy to convince oneself that the optimal choice is $\alpha = \frac{1}{d+2}$ so that by optimizing with respect to the constant r one can make the lower bound to equal approimately $\exp[-k_d t^{\frac{d}{d+2}}]$ with an explicit constant k_d. The precise theorem is

Theorem 1.1. *For any $\nu > 0$,*

$$\liminf_{t\to\infty} \frac{1}{t^{\frac{d}{d+2}}} \log E\big[\exp[-\nu\xi(t)]\big] = k_d(\nu)$$

where $k_d(\nu) > 0$ is explicitly given as

$$\inf_{r>0}[\nu v_d(-\log p) r^{\frac{d}{d+2}} + c_d r^{\frac{2}{d+2}}]$$

Proof. The main step is to provide a bound of the form

$$\log \Pr\big[\text{ the random walk is confined to the ball of radius } \ell \text{ during } [0,t]\ \big] \geq -c_d \frac{t}{\ell^2}$$

where c_d is the Dirichlet ground state eigen value of the properly normalized Laplacian for the unit ball. The idea is that the central limit theorem will allow us to replace the random walk by Brownian motion. Let G be a nice region and let $G_1 \subset G$ be such that $G_1 \subset \bar{G}_1 \subset G$ be compactly contained in G. Let us denote by

$$p(t) = \inf_{x\in G_1} P_x[x(s) \in G \text{ for } 0 \leq s \leq t \text{ and } x(t) \in G_1]$$

From the eigen function expansion

$$p_G(t,x,y) = \sum_j e^{\lambda_j t}\phi_j(x)\phi_j(y)$$

valid for the fundamental solution of the heat equation with Dirichlet boundary conditions

$$p(t) \sim [\inf_{x\in G_1} \phi_1(x)] e^{-\lambda_1 t} \int_{G_1} p(t,x,y)dy$$

By general Markov property $p(nt) \geq p(t)^n$ and the exponential decay rate of $p(t)$ is the groundstate Dirichlet eigenvalue λ_1. Now if we denote by $q_\ell(t,i,j)$the probability of transition from i to j for the random walk remaining inside the ball of radius ℓ during the time $[0,t]$ again by Markov property

$$q(\ell,t) = \inf_{i\in B_{\frac{\ell}{2}}} \sum_{j\in B_{\frac{\ell}{2}}} q_\ell(t,i,j)$$

satisfies

$$q(\ell, nt) \geq q(\ell,t)^n$$

and of course $q(\ell,t)$ is a lower bound for the probability in question. The functional central limit theorem assures us that

$$\lim_{\ell\to\infty} q(\ell, t\ell^2) = p(t)$$

for the suitably normalized Brownian motion. Since $q(\ell,t) \geq [q(\ell, T\ell^2)]^{\frac{t}{T\ell^2}}$, for $t >> \ell^2$ it is easy to establish

$$\lim_{\substack{\ell\to\infty \\ t\to\infty \\ \frac{t}{\ell^2}\to\infty}} \frac{\ell^2}{t} \log \Pr\big[\text{ the random walk is confined to the ball of radius } \ell \text{ during } [0,t]\ \big] \geq -c_d$$

the groundstate eigenvalue of the normalized Laplacian for the unit ball in R^d. $\square$

Now we work towards an upper bound which is harder. We need to estimate $\Pr[\xi(t) \le n]$ from above. For the lower bound we could choose a single set E of cardinality n and estimate $\Pr[x(s) \in E \text{ for } 0 \le s \le t]$ and we simply optimized over suitable choices of E. Although this will provide a reasonable upper bound for each E to get a real upper bound for the probability in question we have to sum over different E's and they are just too many to make the upper obtained in this manner to be meaningful. So we proceed rather carefully to obtain the upper bound using techniques from large deviation theory. Since we are dealing with upper bounds we know that compactness will be a serious issue. We obtain compactness if we replace $Z^d \subset R^d$ by a sublattice $Z^d_N \subset T^d_\ell$ the d-dimensional torus of size ℓ. We will pick Z^d_N to be the sublattice modulo $\ell t^{\frac{1}{d+2}}$ and imbed it inside the torus of size ℓ by scaling the lattice sites by a factor of $h = t^{-\frac{1}{d+2}}$. Let us rescale time by a factor of $h^2 = t^{-\frac{2}{d+2}}$ so that our new random walk has the generator

$$L_h f(x) = \frac{1}{2h^2} \sum_y [f(x + hy) - f(x)]$$

on the torus T^d_ℓ and runs upto time $\tau = t^{\frac{d}{d+2}}$. Since the projection on to any torus reduces the number of distinct sites

$$\Pr[\xi(t) \le n] \le P_{h,\ell}[\xi(\tau) \le n]$$

and we could try to calculate

$$-\theta(\ell) = \limsup_{\tau\to\infty} \frac{1}{\tau} \log P_{h,\ell}[\xi(\tau) \le a\tau]$$

and, it would follow that

$$\limsup_{t\to\infty} \Pr[\xi(t) \le a t^{\frac{d}{d+2}}] \le -\limsup_{\ell\to\infty} \theta(\ell).$$

Step 1. Let us consider a Markov process with generator L_h on the torus T^d_ℓ for times $0 \le t \le \tau$, starting from the origin and view the occupation measure

$$\mu_\tau(A) = \frac{1}{\tau} \int_0^\tau \chi_A(x(s))ds$$

as an element of $\mathcal{M}_\ell$ the space of probability measure on the torus T^d_ℓ. If we denote the process by P_h and the measure induced on $\mathcal{M}_\ell$ by $Q_{h,\tau}$ we want a large deviation upper bound for closed sets $C \subset \mathcal{M}_\ell$,

$$\limsup_{\tau\to\infty} \frac{1}{\tau} \log Q_{h,\tau}[C] \le -\inf_{\mu\in C} I(\mu)$$

where $I(\mu)$ is the rate function for Brownian motion

$$I_\ell(\mu) = \frac{1}{8} \int_{T^d_\ell} \frac{|\nabla f|^2}{f} dx$$

Suppose $U(x)$ is a smooth positive function on the torus, by Feynman-Kac formula

$$\begin{aligned} U(0) &= E^{P_h}\big[\exp[\int_0^\tau \frac{L_h U}{U}(x(s))ds]U(x(\tau))\big] \\ &\geq \frac{\inf_x U(x)}{\sup_x U(x)} E^{Q_{h,\tau}}\big[\exp[\tau \int \frac{L_h U}{U}(x) d\mu(x)]\big] \end{aligned}$$

Although h is related to τ by the relation $h = \tau^{-\frac{1}{d}}$ it plays no role in this calculation. As $h \to 0$, $\frac{L_h U}{U} \to \frac{\Delta U}{2U}$, and this will give by Tchebechev's inequality a local decay rate for $Q_{h,\tau}$ around μ of the form

$$\limsup_{N \downarrow \mu} \limsup_{\tau \to \infty} \frac{1}{\tau} \log Q_{h,\tau}[N] \leq \int \frac{\Delta U}{2U} d\mu$$

It is a simple matter to check that

$$\inf_{U>0} \int \frac{\Delta U}{2U} d\mu = I(\mu).$$

Since the space is compaact this is enough to provide the upper bound. Unfortunately this upper bound as it stands is worthless. The set $\xi(\tau) \leq a t^{\frac{d}{d+2}}$ or $\xi(\tau) \leq a\tau$ is expressed very badly in terms of the cardinality of the support of the occupation measure μ_t. It is very unstable in the weak topolgy for μ.

Step 2. We take the uniform distribution on a small cube of side h which is just enough to cover the gaps between the lattice points and convolute μ with it. More specifically we have a map ϕ_h mapping $\mathcal{M}_\ell$ into $L_1[T_\ell^d]$. If we denote by $Q'_{h,\ell}$ the measure induced on L_1 we would like to show that an LDP holds in L_1 with the same rate function as before namely $I(\mu)$. Once this is done

$$\frac{1}{t^{\frac{d}{d+2}}} \xi(t) = |x : \mu * \phi_h > 0|$$

and the function $|x : f(x) > 0|$ is not a bad function on L_1. It is infact lower semicontinuous in the strong topology of L_1. Therefore we would have that $\{\xi(t) \leq a t^{\frac{d}{d+2}}\} = \{|x : \mu_\tau * \phi_h(x) > 0| \leq a\}$ is a closed set in L_1 (Here $|A|$ is the Lebesgue measure of the set A) there by giving us the upper bound

$$\theta(\ell) = \inf \{I(f) \,||\, f : |x : f(x) > 0| \leq a\}$$

and if we let $\ell \to \infty$ it is not hard to see that the torus gets replaced by R^d. In the end the upper bound involves

$$\inf \{I(f) \,||\, f : |x : f(x) > 0| \leq a\}$$

where the same quantities are now considered on R^d. If we fix the set A, then clearly

$$\inf_{supp f \subset A} I(f) = \lambda_1(A)$$

the First Dirichlet eigenvalue of $-\frac{1}{2}\Delta$ for the region A. We now have to carry out the minimizing of $\lambda_1(A)$ over all A with a given volume. By isoperimetric inequality this is achieved for a ball. The best bet is to take a ball of suitable radius, and if we do that we match the lowerbound because that is how we derived it.

Step 3. The proof of the LDP in L_1. We already have an LDP for $Q_{h,\tau}$ on $\mathcal{M}_\ell$. The question really is if convolution with ϕ_h is enough mollification to get us exponential tightness in L_1. We will assume the following lemma which we will prove later, and show that the proof can be completed modulo the lemma. **Lemma:** Let us consider $Q'_{h,\epsilon}$ on L_1. Then

denoting by by g_ϵ some approximation of identity and by $f_\epsilon = f * g_\epsilon$

$$\limsup_{\epsilon\to 0}\limsup_{\tau\to\infty}\frac{1}{\tau}\log Q'_{h,\tau}\big[f : \|f - f_\epsilon\| \geq \delta\big] = -\infty$$

for every $\delta > 0$. Given the lemma, for any closed set $C \subset L_1$

$$Q'_{h,\tau}[f \in C] \leq Q'_{h,\tau}[f_\epsilon \in C^\delta] + Q'_{h,\tau}[\|f - f_\epsilon\| \geq \delta]$$

Since convolution by g_ϵ is a continuous map from $\mathcal{M}_\ell$ into L_1, if we denote by $\bar{C}^\delta$ the closure of C^δ in L_1

$$\limsup_{\tau\to\infty}\frac{1}{\tau}\log Q'_{h,\tau}[f_\epsilon \in \bar{C}^\delta] \leq k_\epsilon = -\inf_{f: f_\epsilon\in\bar{C}^\delta} I(f)$$

From the lemma the second part will be super-exponentially small and the proof is completed by letting $\epsilon \to 0$. **Proof of Lemma**. We need to evaluate

$$Q_{h,\tau}[\mu : \|\mu * \phi_h * g_\epsilon - \mu * \phi_h\| \geq \delta]$$

We do it in the following manner.

$$\begin{aligned}\|\mu * \phi_h - \mu * \phi_h * g_\epsilon\| &= \sup_{V:|V|\leq 1}\int [V * \phi_h * f_\epsilon - V * \phi_h] d\mu(x)\\ &= \sup_{\substack{W: W=\phi_h * V\\ |V|\leq 1}}\int [W * f_\epsilon - W] d\mu(x)\end{aligned}$$

We would like to find a finite number N of W's such that any other W is with in a distance $\frac{\delta}{4}$ of one of these. Assume we can do that with a choice $N = N(h)$ that we will estimate later. Then

$$\begin{aligned}Q_{h,\tau}[\mu : \|\mu * \phi_h * g_\epsilon - \mu * \phi_h\| \geq \delta] &\leq Q_{h,\tau}[\sup_{1\leq i\leq N}\int [W_i * f_\epsilon - W_i] d\mu(x) \geq \frac{\delta}{2}]\\ &\leq N \sup_{1\leq i\leq N} Q_{h,\tau}[\int [W_i * f_\epsilon - W_i] d\mu(x) \geq \frac{\delta}{2}]\end{aligned}$$

We need therefore to estimate two terms : **Step 1.** Uniformly for W with $|W| \leq b$

$$E_0\big[\exp[\int_0^\tau W(x(s))ds]\big] \le C_b \exp[\tau\lambda_h(W)]$$

for a constant independent of W and h, where $\lambda_h(W)$ is the first eigenvalue of the operator $L_h + W$. By local limit theorem it is easy to get a bound of the form

$$\sup_{x.x'} \frac{p_h(1,x,y)}{p_h(1,x',y)} \le C$$

uniformly as $h \to 0$. Using Markov property

$$\begin{aligned}
E_x\big[\exp[\int_0^\tau W(x(s))ds]\big] &\le e^b E_x\big[\exp[\int_1^\tau W(x(s))ds]\big] \\
&= e^b \int p_h(1,x,dy) E_y\big[\exp[\int_0^{\tau-1} W(x(s))ds]\big] \\
&\le e^b C \int p_h(1,x',dy) E_y\big[\exp[\int_0^{\tau-1} W(x(s))ds]\big] \\
&\le e^{2b} C E_{x'}\big[\exp[\int_0^\tau W(x(s))ds]\big]
\end{aligned}$$

If we denote by

$$U(\tau,x) = E_x\big[\exp[\int_0^\tau W(x(s))ds]\big]$$

then by Markov property $\sup_x U(\tau,x)$ is submultiplicative and $\inf_x U(\tau,x)$ is supermultiplicative. Their ratio is bounded uiformly by C_δ. Their growth is exponential with a constant $\lambda_h(W)$. Therefore

$$\inf_x U(\tau,x) \le \exp[\tau\lambda_h(W)] \le \sup_x U(\tau,x) \le C_b \inf_x U(\tau,x)$$

and we are done. Now we can estimate by Tchebechev's inequality. If we can show that

$$\limsup_{\epsilon\to 0} \sup_{V:|V|\le\sigma} \lambda_h(V - V * g_\epsilon) = 0$$

for every $\sigma > 0$, then the probability will be superexponentially small. By the variational formula

$$\lambda_h(V - V * g_\epsilon) = \sup_\psi \sum_x V(x)[\psi(x) - \psi * g_\epsilon(x)] - I_h(g)$$

and we need only to control the L_1 modulus continuity

$$\sum_x |\psi(x+z) - \psi(x)|$$

in terms of the Dirichlet form

$$\sum_{\substack{x,y\\ x\sim y}} |\sqrt{\psi(x)} - \sqrt{\psi(y)}|^2$$

and this is done by Schwarz's inequality.

Step 2. Now it is just a question of estimating the number $N = N(h)$. The set functions W that we have to deal with are defined on a lattice in T_ℓ^d with $(\ell h)^{-d} = \ell^d \tau$ points in it. So the set in question is the cube of side 2 in $R^{\ell^d \tau}$ and needs $\left(\frac{8}{\delta}\right)^{\ell^d \tau}$ points to populate with in a distance of $\frac{\delta}{2}$ from every W. This is only an exponential growth in τ and does not affect the superexponential decay.