Random time change.

If P is a solution to the martingale problem for

$$L = \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{d} b_(x)\frac{\partial}{\partial x_j}$$

we can make a transformation by rescaling time $y(t) = (T_2 x)(t) = x(2t)$. Then the new process $Q = PT_2^{-1}$ will be a solution to the martingale problem for $2L$. We just need to observe that if $M(t)$ is a martingale with respect $(\mathcal{F}_t, P)$, for any $c > 0$, $M(ct)$ is a martingale with respect $(\mathcal{F}_{ct}, P)$ and all we have done is change the time scale. There is the possibility of changing the time scale differently at different points. Let $c(x)$ be a measurable function that satisfies $0 < c_1 \leq c(x) \leq c_2$, and we run the clock at speed $\frac{1}{c(x)}$ when the trajectory is at x. So at time t the clock shows

$$\sigma_t = \int_0^t \frac{1}{c(x(s))} ds$$

It will show a time of t at time τ_t which is a solution is the solution of

$$\sigma_{\tau_t} = \int_0^{\tau_t} \frac{1}{c(x(s))} ds = t$$

Then τ_t is a stopping time and the random time change

$$y(t) = x(\tau_t)$$

defines a map $\Theta_{c(\cdot)} : C[[0,\infty); R^d] \to C[[0,\infty); R^d]$. It is easy to check that

$$\Theta_{c_1(\cdot)}\Theta_{c_2(\cdot)} = \Theta_{c_1(\cdot)c_2(\cdot)}$$

If $X(t)$ is a martingale with respect to $(\Omega, \mathcal{F}_t, P)$ and τ_t is an increasing family of stopping times then $Y(t) = X(\tau_t)$ is a martingale with respect to $(\Omega, \mathcal{F}_{\tau_t}, P)$. It follows that $Q = P\Theta_{c(\cdot)}^{-1}$ is a solution for

$$(\widehat{L}u)(x) = c(x)(Lu)(x)$$

The steps are reversible and $\Theta_{c(\cdot)}$ is an invertible map with $\Theta_{\frac{1}{c(\cdot)}}$ being the inverse. In particular if $0 < c_1 \leq c(x) \leq c_2 < \infty$, P is a solution for L if and only if $Q = P\Theta_{c(\cdot)}^{-1}$ is a solution for $c(x)L$. Existence or uniqueness for L implies and is implied respectively by Existence or uniqueness for $c(x)L$

This proves in one dimension existence and uniqueness for any

$$L = \frac{1}{2}a(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x}$$

provided $0 < c_1 \leq c(x) \leq c_2 < \infty$ and $|b(x)| \leq C < \infty$.

Special situation in d=1,2.

We consider time dependent operator in 1-d

$$\frac{1}{2}a(t,x)\frac{\partial^2}{\partial x^2}$$

where $a(t,x)$ is measurable and satisfies $0 < c_1 \le a(t,x) \le c_2 < \infty$. We need to be able to solve for $t \le T$

$$\frac{\partial u}{\partial t} + \frac{1}{2}a(t,x)\frac{\partial^2 u}{\partial x^2} = f(t,x); u(T,x) = 0$$

in $W_p^{1,2}$ for some p such that

$$\|u\|_\infty \le C\|u\|_p^{1,2}$$

For the heat equation $u_t + \frac{C}{2}u_{xx} = f$ $p(t,x,y) = \frac{1}{\sqrt{2\pi Ct}}e^{-\frac{(x-y)^2}{2Ct}}$

$$\sup_{0\le s\le T, x\in R}\int_s^T\int_R |p(t-s,x,y)|^2 dtdy = \sup_{0\le s\le T}\int_s^T \frac{1}{2\pi Ct}\ \sqrt{\pi Ct}\ dt = c(T) < \infty$$

We can afford to take $p = 2$ for the perturbation.

If $u_t + \frac{C}{2}u_{xx} = f$ then $i\tau\hat{u}(\tau,\xi) - \frac{C\xi^2}{2} = \hat{f}(\tau,\xi)$. We meed to invert $D_t + \frac{1}{2}a(t,x)D_{xx}$ on $L_2[[0,T]\times R^d]$ and get a $u \in W_2^{1,2}$ on $[[0,T]\times R^d]$. Since $|\frac{\xi^2}{\frac{C}{2}\xi^2 - i\tau}| \le \frac{2}{C}$,

$$\begin{aligned}[D_t + \frac{1}{2}a(t,x)D_{xx}]^{-1} &= [D_t + \frac{C}{2}D_{xx} + \frac{1}{2}[a(t,x) - C]D_{xx}]^{-1} \\ &= [D_t + \frac{C}{2}D_{xx}]^{-1}[I + [\frac{1}{2}[a(t,x)-C]D_{xx}][D_t + \frac{C}{2}D_{xx}]^{-1}]^{-1}\end{aligned}$$

More over $[|\frac{1}{2}[a(t,x) - C]\frac{2}{C}| \le \frac{C-c_1}{C} < 1$. Rest is as before. $p = 2$ works

d=2.

We consider

$$\frac{1}{2}\sum_{i,j=1}^{2} a_{i,j}(x)D_{x_i}D_{x_j}$$

We can do a random time change and assume that the trace of $\{a_{i,j}(x)\} = a_{1,1}(x) + a_{2,2}(x) \equiv 2$ or $(a_{1,1}(x) - 1) = (1 - a_{2,2}(x))$. We also have an estimate of the form

$$\|u\|_\infty \le C|\lambda u - \frac{\Delta}{2}u\|_2$$

because

$$u(x) = \int \frac{e^{-i<x,\xi>}}{(\lambda + \frac{\xi^2)}{2}}\hat{f}(\xi)d\xi$$

and $(\lambda + \frac{\xi^2}{2})^{-2}$ is integrable. It is enough to work on L_2. For some $\rho > 0$ and $c > 0$ $|a_{1,2}(x)|^2 \le (1-\rho)a_{1,1}(x)a_{2,2}(x)$ and $a_{1,1}(x) \ge c$ and $a_{2,2}(x) \ge c$.

$$
\begin{aligned}
&\|[a_{1,1}(x)-1]u_{xx} + 2a_{1,2}u_{xy} + [a_{2,2}(x)-1]u_{yy}\|_2^2 \\
&= \|[a_{1,1}(x)-1][u_{xx}-u_{yy}] + 2a_{1,2}(x)u_{xy}\|_2^2 \\
&\le \int [[a_{1,1}(x)-1]^2 + a_{1,2}^2(x)][(u_{xx}-u_{yy})^2 + 4u_{xy}^2]dxdy \\
&\le \sup_x [[a_{1,1}(x)-1]^2 + (1-\rho)a_{1,1}(x)a_{2,2}(x)] \int [(u_{xx}-u_{yy})^2 + 4u_{xy}^2]dxdy \\
&\le \sup_x [(a_{1,1}(x)-1)^2 + a_{1,1}(x)(2-a_{1,1}(x)] - \rho a_{1,1}(x)a_{2,2}(x)] \\
&\qquad\qquad\qquad\qquad \times \int [(u_{xx}-u_{yy})^2 + 4u_{xy}^2]dxdy \\
&\le (1-\rho c^2) \int [(u_{xx}-u_{yy})^2 + 4u_{xy}^2]dxdy \\
&= (1-\rho c^2) \int [(\xi^2-\eta^2)^2 + 4\xi^2\eta^2]|\hat{f}(\xi,\eta)|^2]d\xi d\eta \\
&= (1-\rho c^2)\|\widehat{\Delta f}\|_2^2
\end{aligned}
$$

In **1**-d, the S.D.E

$$dx(t) = \sigma(x(t))d\beta(t)$$

has a unique sloution provided σ is Hölder continuous of exponent $\frac{1}{2}$. If we have two solutions, $x(t)$ and $y(t)$, $z(t) = x(t) - y(t)$ stsfies

$$dz(t) = \int_0^t [\sigma(x(s)) - \sigma(y(s))]d\beta(s)$$

We take a function $\phi(z) \ge 0$, $\phi(0) = 0$ that is twice differentiable and

$$E[\phi(z(t))] = \frac{1}{2}E\left[\int_0^t [\sigma(x(s)) - \sigma(y(s))]^2 \phi_\epsilon''(z(s))ds\right]$$

Take $\phi_\epsilon(z) = (\epsilon^2 + z^2)^{\frac{1}{2}}$ and let $\epsilon \to 0$. LHS goes to $E[|z(t)|]$.Since $|\sigma(x) - \sigma(y)|^2 \le C|x-y|$ one checks that $|\phi_\epsilon''(z)|z| \to 0$ pointwise and is uniformly bounded. Bounded convergence theorem completes the proof.