Suppose $u$ is a smooth function on $R^d$ with compact support or decays fast enough and $\Delta u = f$, according to a theorem of Calderon and Zygmund, for $1 < p < \infty$ there is a constant $C(d,p)$ such that

$$\|\frac{\partial^2 u}{\partial x_i \partial x_j}\|_p \le C(d,p)\|f\|_p$$

The heat kernel in $d$ dimension is given by.

$$p(t,x,y) = \frac{1}{(2\pi)^{\frac{d}{2}}} \exp[\frac{(x-y)^2}{2t}]$$

the semigroup by

$$(T_t f)(x) = \int_{R^d} f(y)p(t,x,y)dy$$

and $R_\lambda$ the resolvent by

$$u(x) = (R_\lambda f)(x) = \int_0^\infty \int_{R^d} e^{-\lambda t} f(y)p(t,x,y)dydt = \int_0^\infty e^{-\lambda t}(T_t f)(x)dt$$

It solves the equation

$$\lambda u - \frac{1}{2}\Delta u = f$$

Since $\|T_t f\| \le \|f\|_p$ for every $t > 0$ and $1 \le p \le \infty$, $\|\lambda R_\lambda f\|_p \le \|f\|_p$. $\Delta u = 2(\lambda u - f)$ and $\|\Delta u\|_p \le 4\|f\|_p$. In particular for $1 < p < \infty$ there is a constant $C(p,d)$ such that for all $f \in L_p(R^d)$, $i,j = 1,\ldots,d$ and $\lambda > 0$,

$$\|D_{x_i} D_{x_j} R_\lambda f\|_p \le C(p,d)\|f\|_p$$

Let $\epsilon(p,d) = \frac{1}{d^2 C(p,d)}$. Then if $\sup_{i,j,x} |a_{i,j}(x) - \delta_{i,j}| \le \epsilon(p,d)$

$$\left|(\lambda R_\lambda f)(x) - \frac{1}{2}\sum_{i,j} a_{i,j}(x)\frac{\partial^2 R_\lambda f}{\partial x_i \partial x_j}(x) - f(x)\right| \le \left|\frac{1}{2}\sum_{i,j}[[a_{i,j}(x) - \delta_{i,j}]\frac{\partial^2 R_\lambda f}{\partial x_i \partial x_j}(x)\right|$$
$$\le \frac{1}{2}\sup_{i,j,x}|a_{i,j}(x) - \delta_{i,j}|\sum_{i,j}|\frac{\partial^2 R_\lambda f}{\partial x_i \partial x_j}(x)|$$

$$\left|(\lambda R_\lambda f)(\cdot) - \frac{1}{2}\sum_{i,j} a_{i,j}(\cdot)\frac{\partial^2 R_\lambda f}{\partial x_i \partial x_j}(\cdot) - f(\cdot)\right|_p \le \frac{1}{2}\epsilon(p,d)d^2 C(d.p)\|f\|_p = \frac{1}{2}\|f\|_p$$

Denoting by

$$L = \frac{1}{2}\sum_{i,j} a_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}$$

and by

$$L_0 = \frac{1}{2}\Delta = \frac{1}{2}\sum_i \frac{\partial^2}{\partial x_i^2}$$

$$\|\lambda R_\lambda - LR_\lambda - I\|_{p\to p} \le \frac{1}{2}$$

Implies $\lambda R_\lambda - LR_\lambda$ is an invertible operator $L_p \to L_p$. In particular since $R_\lambda$ maps smooth functions into smooth functions, for any $\lambda > 0$ the range of $\lambda u - Lu$ as $u$ varies over smooth functions with compact support is dense in $L_p$.

If $P_1, P_2$ are two solutions on $(C[[0,T]; R^d, \mathcal{F}_t)$ with $P_i[x(0) - x_0] = 1$ that make

$$X_u(t) = u(x(t)) - u(x(0)) - \int_0^t (Lu)(x(s))ds$$

a martingale for smooth $u$, then for smooth $u$

$$e^{-\lambda t} u(x(t)) - u(x(0)) - \int_0^t e^{-\lambda s}(\lambda u - Lu)(x(s))ds$$

is a martingale and

$$E^{P_i}[\int_0^\infty e^{-\lambda s}(\lambda u - Lu)(x(s))ds] = u(x_0)$$

We want to conclude that

$$E^{P_1}[\int_0^\infty e^{-\lambda s} f(x(s)ds] = E^{P_2}[\int_0^\infty e^{-\lambda s} f(x(s)ds]$$

for sufficiently many $f$ and therefore by the uniqueness result for Laplace transforms conclude that

$$E^{P_1}[f(x(t)] = E^{P_2}[f(x(t)]$$

Since the set of functions $f$ in the range of $\lambda I - L$ is dense in $L_p$ we need to show that for any solution $P$ the probability measure

$$\mu(A) = \int_0^\infty e^{-\lambda t} P[x(t) \in A]dt$$

is in $L_q$ where $\frac{1}{p} + \frac{1}{q} = 1$. For Brownian motion the singularity at the origin is $\simeq |x|^{2-d}$. For $\lambda > 0$ there is exponential decay at $\infty$ so that $\mu \in L_q$ if $q(d-2) < d$ or $q < \frac{d}{d-2}$or $p > \frac{d}{2}$. Fix $p_0 > \frac{d}{2}$. If $P$ is a solution for

$$L = \frac{1}{2}\sum_{i,j} a_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}$$

and

$$|a_{i,j}(x) - \delta_{i,j}| \le \epsilon(p_0, d)$$

so that for any $\lambda > 0$, the range of $\lambda u - Lu$ as $u$ ranges over smooth functions with compact support is dense in $L_{p_0}$. Let $q_0$ satisfy $p_0^{-1} + q_0^{-1} = 1$. Let $P$ be a solution for $L$ with $P[x(0) = x_0] = 1$. Then we know that if $f = \lambda u - Lu$,

$$u(x_0) = E^P[\int_0^\infty e^{-\lambda s} f(x(s))ds] = \int f(x)\mu_\lambda(dx)$$

where $\mu_\lambda(A) = \int_0^\infty e^{-\lambda t} P[x(t) \in A]dt$.

$$\begin{aligned}
[(\lambda I - L_0)^{-1} f](x_0) &= E^P \int_0^\infty e^{-\lambda t}[(\lambda I - L)(\lambda - L_0)^{-1} f](x(t)dt \\
&= E^P \int_0^\infty e^{-\lambda t}[(\lambda I - L_0 + L_0 - L)(\lambda - L_0)^{-1} f](x(t)dt \\
&= E^P \int_0^\infty e^{-\lambda t}[[I - (L - L_0)(\lambda - L_0)^{-1}]f](x(t)dt \\
&= < f, \mu_\lambda > - < (L - L_0)(\lambda - L_0)^{-1} f, \mu_\lambda >
\end{aligned}$$

Let us take supremum over $f$ with $\|f\|_p \le 1$. Then

$$\|\mu_\lambda\|_q \le c(q, d) + \epsilon(p, d)\|\mu_\lambda\|_q$$

Either $\|\mu_\lambda\|_q = \infty$ or $\|\mu_\lambda\|_q \le (1 - \epsilon(p, d))^{-1} c(p, d)$ We have an a priori bound, but we need to approximate $P$ by $P_n$ satisfying the same bound and pass to the limit. We know that there is a BM such that with $\sigma\sigma^* = a$, $\sigma = a^{\frac{1}{2}}$.

$$x(t) = x_0 + \int_0^t \sigma(x(s)) \cdot d\beta(s)$$

we can approximate by

$$x^n(t) = x_0 + \int_0^t \sigma^n(s, \omega) \cdot d\beta(s)$$

where $a^n(s, \omega) = \sigma^n(s, \omega)\sigma^n(s, \omega)^*$ satisfies $d^2 \sup_{i,j,t,\omega} |a^n_{i,j}(t, \omega) - \delta_{i,j}| \le \epsilon(p, d)$.

Let $\mu^n_\lambda(A) = \int_0^\infty e^{-\lambda t} P[x^n(t) \in A]dt$.

$$[(\lambda I - L_0)^{-1} f](x_0) = E^P \int_0^\infty e^{-\lambda t}[(\lambda I - \frac{1}{2}\sum_{i,j} a^n_{i,j}(t, \omega)\frac{\partial^2}{\partial x_i \partial x_j}))(\lambda - L_0)^{-1} f](x^n(t)dt$$

$$= E^P \int_0^\infty e^{-\lambda t}[(\lambda I - L_0 + L_0 - \frac{1}{2}\sum_{i,j} a^n_{i,j}(t, \omega)\frac{\partial^2}{\partial x_i \partial x_j})(\lambda - L_0)^{-1} f](x^n(t)dt$$

$$= E^P \int_0^\infty e^{-\lambda t}[I - \frac{1}{2}\sum_{i,j}(a^n_{i,j}(t, \omega) - \delta_{i,j})\frac{\partial^2}{\partial x_i \partial x_j})(\lambda - L_0)^{-1}]f](x^n(t)dt$$

$$= < f, \mu^n_\lambda > - < \frac{1}{2}\sum_{i,j}(a^n_{i,j}(t, \omega) - \delta_{i,j})\frac{\partial^2}{\partial x_i \partial x_j}))(\lambda - L_0)^{-1} f, \mu^n_\lambda >$$

Let us again take supremum over $f$ with $\|f\|_p \le 1$. Then

$$\|\mu^n_\lambda\|_q \le c(q,d) + \epsilon(p,d)\|\mu^n_\lambda\|_q$$

But $\|\mu_\lambda\|_q < \infty$ and $\|\mu_\lambda\|_q \le (1-\epsilon(p,d))^{-1}c(p,d)$ We can now pass to the limit.

A similar argument holds for time dependent situation where we have $a_{i,j}(t,x)$ and then the Laplace transform is not useful. We need to solve the Cauchy problem

$$u_t + \frac{1}{2}\sum_{i,j} a_{i,j}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j} = -f(t,x); \text{ for } s < T, \text{ and } u(T,x) = 0$$

Then if $P$ is a solution with $P[x(s) = x_0] = 1$,

$$u(t,x(t)) - u(s,x(s)) - \int_s^t u_\sigma(\sigma,x(\sigma))d\sigma - \int_s^t \frac{1}{2}\sum_{i,j} a_{i,j}(\sigma,x(\sigma))\frac{\partial^2 u}{\partial x_i \partial x_j}(\sigma,x(\sigma))d\sigma$$

l.e.

$$u(t,x(t)) - u(s,x(s)) + \int_s^t f(\sigma,x(\sigma))d\sigma$$

is a martingale with respect to any solution $P$ with $P[x(s) = x_0] = 1$ . Equating expectations at $t = s$ and $t = T$

$$u(s,x_0) = E^P[\int_s^T f(\sigma,x(\sigma))d\sigma]$$

If enough expectations are determined, then $P$ is unique. The equation

$$u_t + \frac{1}{2}\Delta u = -f; u(T,x) = 0$$

has the solution

$$u(s,x) = \int_s^T \int_{R^d} f(t,y)\frac{1}{(2\pi(t-s))^{\frac{d}{2}}}\exp[-\frac{(y-x)^2}{2(t-s)}]dydt$$

One has analogs of Calderon-Zygmund estimates (Ben Franklin Jones) that estimate $\|u_t\|_p, \|u_{x_i,x_j}\|_p$ on $R^d \times [0,T]$ in terms of $\|f\|_p$ on $R^d \times [0,T]$ for $1 < p < \infty$. This allows the perturbation to work. We need to pick a $p$ such that for $q = \frac{p}{p-1}$

$$\int_0^T \int_{R^d} \frac{1}{(2\pi t)^{\frac{dq}{2}}}\exp[-\frac{qy^2}{2t}]dydt < \infty$$

i.e $d(q-1) < 2$ or $q < 1 + \frac{2}{d}$ or $p > \frac{1+\frac{2}{d}}{\frac{2}{d}} = \frac{d+2}{2}$