We will consider $a(x) = \{a_{i,j}(x)\}$ and $b(x) = \{b_j(x)\}$ that do not depend on t explicitly, It is not serious at this point. We can consider time as an extra space variable and add one extra dimension.

$$\hat{a}(x_0, x) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & a_{1,1}(x_0, x) & a_{1,2}(x_0, x) \cdots & a_{1,d}(x_0, x) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & a_{d,1}(x_0, x) & a_{d,2}(x_0, x) \cdots & a_{d,d}(x_0, x) \end{pmatrix}$$

$$\hat{b}(x_0, x) = (1, b_1(x_0, x), \ldots, b_d x_0, (x))$$

So long as no non-degeneracy of a is needed.

Given $a(\cdot), b(\cdot)$ we denote by $\mathcal{S}(a, b, x)$ the set of probability measures P on $C[0, \infty)$ that satisfy $P[x(0) = x] = 1$ and

$$f(x(t) - f(x(0)) - \int_0^t (Lf)x(s)ds$$

is a martingale for all smooth C^∞ functions f with compact support where

$$(Lf)(x) = \frac{1}{2} \sum_{i,j=1}^{d} a_{i,j}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{j=1}^{d} b_j(x) \frac{\partial f}{\partial x_j}(x)$$

What do we know?

1. If $a(\cdot), b(\cdot)$ are bounded and continuous $\mathcal{S}(a(\cdot), b(\cdot), x)$ is nonempty for every $x \in R^d$.

2. If $\hat{b}(x) = b(x) + a(x)c(x)$ for a bounded $c(\cdot)$ then with

$$Y(t) = \exp[\int_0^t \langle c(x(s)), dx(s) - b(x(s))ds \rangle - \frac{1}{2} \int_0^t \langle c(x(s)), a(x(s))c(x(s)) \rangle ds]$$

is a martingale with respect to $(\Omega, \mathcal{F}_t, P)$ for all $P \in \mathcal{S}(a(\cdot), b(\cdot), x)$ and the map $P \to Q$ defined by $dQ = Y(t)dP$ sets up an isomorphism between $P \in \mathcal{S}(a(\cdot), b(\cdot), x)$ and $Q \in \mathcal{S}(a(\cdot), b(\cdot) + a(\cdot)c(\cdot), x)$. In particular existence or uniqueness for $[a(\cdot), b(\cdot)]$ implies the same for $[a(\cdot), b(\cdot) + a(\cdot)c(\cdot)]$ for any bounded $c(\cdot)$. With a little extra work one can extend it to $[a(\cdot), b(\cdot) + \sqrt{a}(\cdot)c(\cdot)]$ which is the same as $[a(\cdot), b(\cdot) + \sigma(\cdot)c(\cdot)]$ with a bounded $c(\cdot)$ for some $\sigma(\cdot)$ with $\sigma(\cdot)\sigma^*(\cdot) = a(\cdot)$. Note that if $Y(t)$ is a P martingale and $Y(t) > 0$ a.e. P, the $Z(t) = [Y(t)]^{-1}$ is Q martingale and $dP = Z(t)dQ$.

3. if $P \in \mathcal{S}(a, b, x)$ and $P_{t,\omega}$ is the r.c.p.d. $P|\mathcal{F}_t$ then $P_{t,\omega} \in P \in \mathcal{S}(a, b, x(t, \omega))$ for almost all ω for times $s \geq t$. The same is true if we replace t by a stopping time τ. In particular if we have uniqueness, i.e for every x, $\mathcal{S}(a, b, x)$ consists of one distribution P_x, then P_x is a strong Markov process with transition probability

$$p(t, x, A) = P_x[x(t) \in A]$$

4. Stability. If $[a_n(\cdot), b_n(\cdot)]$ converges to $[a(\cdot), b(\cdot)]$ uniformly on compact subsets and are uniformly bounded, and $x_n \to x$ then $\mathcal{S}(a_n, b_n, x_n)$ is a (pre) compact subset of measures and if P is any limit along a subsequence then $P \in \mathcal{S}(a, b, x)$.

5. Let $\pi_h(x, dy)$ be the transition probability of a Marko Chain in R^d with time step h. Assume

$$\lim_{h\to 0} \frac{1}{h} \int [f(y) - f(x)]\pi_h(x, dy) = (Lf)(x) = \frac{1}{2} \sum_{i,j=1}^{d} a_{i,j}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{j=1}^{d} b_j(x) \frac{\partial f}{\partial x_j}(x)$$

Then the piecewise linear interpolated measures $P_{h,x}$ are precompact and any limit point P will be in $\mathcal{S}(a, b, x)$. In particular if $\mathcal{S}(a, b, x)$ consists of a single distribution P_x,

$$\lim_{\substack{n\to\infty, h\to 0\\ nh\to t}} \int f(y)\pi_h^n(x, dy) = (T_t f)(x) = \int f(y)p(t, x, dy) = E[f(x(t)|x(0) = x]$$

6. The following are necessary and for sufficient for the assumption of **5.** to hold.

$$\lim_{h\to 0} \frac{1}{h} \pi_h(x, [B(x, \epsilon)]^c) = 0$$

$$\lim_{h\to 0} \frac{1}{h} \int_{B(x,1)} (y - x)\pi_h(x, dy) = b(x)$$

$$\lim_{h\to 0} \frac{1}{h} \int_{B(x,1)} \int (y - x) \otimes (y - x)\pi_h(x, dy) = a(x)$$

The first limit should hold for any $\epsilon > 0$ and all limits hold locally uniformly. In the last two limits $B(x.1)$ can be replaced by any ball of finite radius.

Proof of **6.** Taylor expansion.

5. Requires some work. Tightness estimates are needed. For the moment assume that the limits in **6** hold uniformly on R^d.

Construct a piecewise linear random function with $x(0) = x$ and $x(nh) = X_n$ and $x(t)$ is linearly interpolated between $x(nh)$ and $(x(n + 1)h)$. Let $P_{h,x}$ be its distribution.

$$f(X_{nh}) - f(x) - \sum_{j=0}^{n-1} (\pi_h f - f)(X_{jh})$$

is a martingale relative to $(\Omega, \mathcal{F}_{nh}, P_{h,x})$.

$$E[f(x(nh)) - f(x) - \sum_{j=0}^{n-1} (\pi_h f - f)(x(jh))|\mathcal{F}_{(n-1)h}]$$

$$= f(x((n - 1)h)) - f(x) - \sum_{j=0}^{n-2} (\pi_h f - f)(x(jh))$$

The expression is a Riemann sum approximation and seen to converge to

$$f(x(t)) - f(x) - \int_0^t (Lf)(x(s))ds$$

If $P_{h,x}$ has a limit point P it will be in $\mathcal{S}(a(\cdot), b(\cdot), x)$.

To prove tightness we need to estimate the modulus of continuity. Starting from x how long does it take for the chain to escape $B(x, \epsilon)$? Let us define a sequence of stopping times. Given ϵ and h, we define

$$\tau_1 = h \inf j : x(jh) \in [B(x, \epsilon)]^c$$

and inductively

$$\tau_{k+1} = h \inf j : jh \geq \tau_k, x(jh) \in [B(x(\tau_k), \epsilon)]^c$$

If X_j never leaves the ball $B(x(\tau_r), \epsilon)$ at some stage we take $\tau_{r+1} = T$. We proceed until for some k, $\tau_{k+1} = T$. This will happen eventually, definitely before $\frac{T}{h}$. The interval $[0, T]$ is now divided into sub intervals

$$0 = \tau_0 < \tau_1 < \cdots < \tau_k < \tau_{k+1} < \cdots < \tau_{k_0} < \tau_{k_0+1} = T$$

Let us denote by δ^* the length of the smallest interval excluding the last one, i.e.

$$\delta^* = \inf_{1 \leq r \leq k_0} [\tau_r - \tau_{r-1}]$$

If $|t-s| \leq \delta^*$, the points s and t will be at most in two adjacent intervals and the oscillation $|x(t) - x(s)|$ of the function is at most 2ϵ. If we can get a uniform estimate on $P_{h,x}[\delta^* \leq \delta]$ we can then estimate the modulus of continuity. Let f be nonnegative, with $f = 1$ out side $B(x, \epsilon)$ and $f(x) = 0$. If we stop at time τ_1, since

$$f(x(nh)) - f(x) - \sum_{j=0}^{n-1} (\pi_h f - f)(x(jh))$$

is a martingale

$$E^{P_{h,x}}\Big[f(x(\tau_1))] - \sum_{j=0}^{k-1} (\pi_h f - f)(x(jh))\Big] = 0$$

Since $\|(\pi_h f) - f\| \leq C(\epsilon)h$ and if $\tau_1 < T$, then $f(x(\tau_1)) = 1$ and

$$P_{h,x}[\tau_1 \leq \delta] \leq E^{P_{h,x}}\big[f(x)\big] \leq C(\epsilon)\delta$$

The argument works for any $\tau_{r+1} - \tau_r$ as long as $r < k_0$. The estimates are conditionally uniform. None of them is too small. If we can show that k_0 can not be too large then δ^* can not be too small.

$$P_{h,x}[\delta^* \leq \delta] \leq k P_{h,x}[\tau_1 \leq \delta] + P_{h,x}[k_0 \geq k]$$

$$P_{h,x}[k_0 \geq k] \leq P_{h,x}[\tau_k \leq T] \leq e^T E^{P_{h,x}}[e^{-\tau_k}] \leq e^T [E^{P_{h,x}}[e^{-\tau_1}]]^k$$

If the estimates only hold locally, we can stop the chain when it gets out of a ball of radius ℓ. This family will be tight and any limit point will satisfy the matingale relation until the stopping time $\tau_l = \inf\{t : |x(t)| \geq \ell\}$. The proof is completed by the following

Lemma. Let the processes P_n on $C[[0,T]; R^d]$ when stopped at the exit time from a ball of radius ℓ, $\tau_\ell = \inf\{t : |x(t)| \geq \ell\}$ converge to a limit P^ℓ and

$$f(x(\tau_\ell \wedge t)) - f(x(0)) - \int_0^{\tau_\ell \wedge t} (Lf)(x(s))ds$$

be a martingale for smooth f with respect to P^ℓ. Then P_n is tight and P_n converges to a limit P and with respect to $(C[[0,T]; R^d], \mathcal{F}_t, P)$

$$f(x(t)) - f(x(0)) - \int_0^t (Lf)(x(s))ds$$

is a martingale.

Proof. Need to get an estimate

$$\lim_{\ell \to \infty} \sup_n P_n[\sup_{0 \leq s \leq T} \|x(s)\| \geq \ell] = 0$$

Let if possible along a subsequnce, for some $\ell_n \to \infty$

$$P_n[\sup_{0 \leq s \leq T} \|x(s)\| \geq \ell_n] \geq p > 0$$

Then for every ℓ, the stopped process P_n^ℓ satisfies

$$P^\ell[\sup_{0 \leq s \leq T} \|x(s)\| \geq \ell] = \limsup_{n \to \infty} P_n^\ell[\sup_{0 \leq s \leq T} \|x(s)\| \geq \ell] = \limsup_{n \to \infty} P_n[\sup_{0 \leq s \leq T} \|x(s)\| \geq \ell] \geq p$$

Since

$$f(x(t)) - f(x(0)) - \int_0^t (Lf)(x(s))ds$$

is a martingale with respect to P^ℓ it follows that

$$\lim_{\ell \to \infty} P^\ell[\sup_{0 \leq s \leq T} \|x(s)\| \geq \ell] = 0$$

PDE proof. Let the equation

$$\frac{\partial u(t,x)}{\partial t} = (Lu)(t,x); \quad u(0,x) = f(x)$$

have a $C^{1,2}$ solution. Then with $u_h(j,x) = (\pi_h^j f)(x)$, if $nh \to t$

$$\|u_h(nh,\cdot) - u(t,\cdot)\| \to 0$$

Let us estimate as $h \to 0$

$$\begin{aligned}\int u(jh,y)\pi_h(x,dy) - u(j+1)h,x) \\ &= \int [u(jh,y) - u(jh,x)]\pi_h(x,dy) - [u((j+1)h,x) - u(j,h)] \\ &= h(Lu)(jh,x) - hu_t(jh,x) + o(h) \\ &= o(1)\end{aligned}$$

Adding the telescoping sum $no(h) \to 0$.

Actually if $u(t,x)$ solves $u_t = Lu$, then for any T

$$\frac{\partial u}{\partial t}(T-t,x) + (Lu)(T-t,x) = -u_t(T-t,x) + (Lu)(T-t,x) = 0$$

making $u(T-t,x(t))$ a martingale with respect to every $P \in \mathcal{S}(a(\cdot),b(\cdot),x)$ Equating expectations

$$E^P[x(T)] = u(T,x)$$

for all $P \in \mathcal{S}(a(\cdot),b(\cdot),x)$.

If equations can be solved for enough initial data then $P[x(t) \in A] = p(t,x,A)$ is determined for all $P \in \mathcal{S}(a(\cdot),b(\cdot),x)$. Since the conditional distribution $P|\mathcal{F}_t \in \mathcal{S}(a(\cdot),b(\cdot),x(t))$ for times s larger than t, the conditional distribution of $x(t+s)$ depends only on $x(t)$ and is given by $p(s,x(t)A)$. Sufficiently many smooth solutions of PDE implies uniqueness and Markov property. Turning the agument around existence of a $P \in \mathcal{S}(a(\cdot),b(\cdot),x)$ proves uniquenss of a solution to the PDE. This is meaningful when a and b are unbounded. We will visit this issue later.

There could be other ways of proving uniqueness. That will still enable us to construct the process and prove limit theorems.