1. What is a stochastic process. Stochastic means random. Process is evolution. A stochastic process is some thing that evolves randomly over time. Diffusion process refers to processes that evolve continuously (no discontinuities or jumps)

2. Time can be continuous or discrete. A stochastic process in discrete time is a random sequence. There is (Ω, Σ, P) and a map $\xi : \Omega \to \mathcal{X}$ the space of sequences.$\xi = \{X_j\}$. Or simply a measure μ on $\mathcal{X}$. In the continuous case $\mathcal{X}$ is the space of functions. A stochastic process in continuous time is a random function $\xi = x(t, \omega)$ defined on some (Ω, Σ, P) or a measure on $\mathcal{X} = \mathcal{F}[a, b]$. For studying diffusion processes $\mathcal{F}[a, b] = C[a, b]$.

3. Brownian motion on $[0, T]$ is the canonical diffusion process. (Ω, Σ, P) What do we want of $\xi(t, \omega)$? For every n, with $x_0 = 0$ and $0 = t_0 < \cdots < t_n \leq T$

$$P[(\xi(t_1, \omega), \ldots, \xi(t_n, \omega)) \in A] = \int_A f_n(t_1, x_1, \ldots, t_n, x_n) dx_1 \cdots dx_n$$

where

$$f_n(t_1, x_1, \ldots t_n, x_n) = c \exp[-\frac{1}{2} \sum \frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}}] = \Pi p(t_{j-1} j, x_{j-1}, t_j, x_j)$$

and $t_0 = x_0 = 0$ and $0 < t_0 < \cdots < t_n \leq T$.

$$p(s, x, t, y) = \frac{1}{\sqrt{2\pi(t - s)}} \exp[-\frac{(x - y)^2}{2(t - s)}]$$

Remark: To be consistent

$$\int f_n(t_1, x_1, \ldots, t_j, x_j, \ldots t_n, x_n) dx_j = f_{n-1}(t_1, x_1, \ldots, t_{j-1}, x_{j-1}, t_{j+1}, x_{j+1}, \ldots t_n, x_n)$$

Reduces to

$$\int p(s, x, t, y) p(t, y, u, z)) dy = p(s, x, u, z)$$

4. Does it exist? The question is equivalent to is there a measure μ on $C[0, T]$ such that for all n and for all $0 < t_0 < \cdots < t_n \leq T$,

$$P[(\xi(t_1, \omega), \ldots, \xi(t_n, \omega)) \in A] = \int_A f(t_1, x_1, \ldots, t_n, x_n) dx_1 \cdots dx_n$$

4.1. Theorem (Kolmogorov). If a consistent family $\{f_n\}$ satisfies for some $\alpha > 0$, $\beta > 0$ and constant C

$$\int |x_1 - x_2|^\beta f_2(t_1, x_1, t_2, x_2) dx_1 dx_2 \leq C |t_1 - t_2|^{1+\alpha}$$

for $0 \leq t_1 < t_2 \leq T$, then μ exists and is unique.

Remark. For BM with $\beta = 4, \alpha = 1$ and $C = 3$ works.

$$\int |x(t_1) - x(t_2)|^4 f_2(t_1.x_1, t_2, x_2) dx_1 dx_2 = \int x^4 p(t_2 - t_1, x) dx = 3|t_1 - t_2|^2$$

Proof of theorem. Let us take $T = 1$ and divide the interval into $n = 2^N$ sub intervals with end points $\frac{j}{2^N}$. On the space $\Omega = R^Z$ of sequences $x(\frac{j}{2^N})$ there is a measure P such that

$$P[(\xi(t_1, \omega), \dots, \xi(t_n, \omega)) \in A] = \int_A f(t_1, x_1, \dots, t_n, x_n) dx_1 \cdots dx_n$$

if $\{t_j\}$ are diadics. For every N, consider the map $\xi_N(t, \omega) : \Omega \to C[0, 1]$ with $\xi_N(\frac{j}{2^N}) = x(\frac{j}{2^N})$ for $0 \le j \le 2^N$ and interpolates linearly over intervals $[\frac{j}{2^N}, \frac{j+1}{2^N}]$. For some $\delta > 0$ we will show that

$$\sum_N P[\sup_t |\xi_{N+1}(t, \omega) - \xi_N(t, \omega)| \ge 2^{-N\delta}] < \infty$$

which is enough. By Borel-Cantelli lemma $\xi(t, \omega) = \lim_{N\to\infty} \xi_N(t, \omega)$ will exist uniformly in t for almost all ω. This defines the limit $\xi(t)$ as a map into $C[0, 1]$ and the image is μ. We note that

$$\sup_{\frac{j}{2^N} \le t \le \frac{j+1}{2^N}} |\xi_N(t, \omega) - \xi_{N+1}(t, \omega)| = |\xi_{N+1}(\frac{2j+1}{2^{N+1}}, \omega) - \frac{1}{2}[\xi_N(\frac{j}{2^N}, \omega) + \xi_N(\frac{j+1}{2^N}, \omega)]|$$

$$\le \max[|\xi_{N+1}(\frac{2j+1}{2^{N+1}}) - \xi_{N+1}(\frac{2j}{2^{N+1}})|, |\xi_{N+1}(\frac{2j+1}{2^{N+1}} - \xi_{N+1}(\frac{2j+2}{2^{N+1}})]$$

$$P[\sup_{\frac{j}{2^N} \le t \le \frac{j+1}{2^N}} |\xi_N(t, \omega) - \xi_{N+1}(t, \omega)| \ge 2^{-(N+1)\delta}] \le 2 \cdot 2^{(N+1)\beta\delta} \cdot 2^{-(N+1)(1+\alpha)} \cdot C$$

$$P[\sup_{0 \le t \le 1} |\xi_N(t, \omega) - \xi_{N+1}(t, \omega)| \ge 2^{-(N+1)\delta}] \le 2^N \cdot 2 \cdot 2^{(N+1)\beta\delta} \cdot 2^{-(N+1)(1+\alpha)} \cdot C$$
$$= 2^{-(N+1)(\alpha - \beta\delta)}$$

Choose $\delta < \frac{\alpha}{\beta}$.

4.2. Garsia-Rodemick-Rumsey theorem.

Let f be continuous on $[0, T]$.Let ψ and p be continuous, even nonnegative functions on R, with $\psi(0) = p(0) = 0$ that are strictly increasing on R^+. $\psi(x) \to \infty$ as $|x| \to \infty$. Assume

$$\int_0^T \int_0^T \psi\left(\frac{|f(t) - f(s)|}{p(t-s)}\right) dt ds = B < \infty$$

Then

$$|f(t) - f(s)| \le 8 \int_0^{|t-s|} \psi^{-1}\left(\frac{4B}{u^2}\right) du$$

With $\psi(x) = |x|^m$ and $p(u) = |u|^a$ for Brownian motion we can get up to Holder $\frac{1}{2} - \delta$ with suitable choices of m and a. $E[B] < \infty$ if

$$\frac{m}{2} - am + a - 1 > -1$$

or

$$a(1-m) + \frac{m}{2} > 0$$

or

$$a < \frac{m}{2(m-1)}$$

We get Holder with $\alpha = a - \frac{2}{m}$. Large m gets α close to $\frac{1}{2}$.

Proof of GRR inequality. Scales correctly. f on $[a,b]$ satisfies.

$$\int_a^b \int_a^b \psi\left(\frac{|f(t) - f(s)|}{p(t-s)}\right) dtds = B$$

$t' = \frac{t-a}{b-a}$ and $s' = \frac{s-a}{b-a}$

$$\begin{aligned} B =& (b-a)^2 \int_0^1 \int_0^1 \psi\left(\frac{|f(a + (b-a)t') - f(a + (b-a)s')|}{p((b-a)(t'-s'))}\right) dt'ds' \\ =& (b-a)^2 \int_0^1 \int_0^1 \psi\left(\frac{|f'(t) - f'(s)|}{p'(t-s)}\right) dtds \end{aligned}$$

with $f'(t) = f(a + (b-a)t)$ and $p'(t) = p((b-a)t)$.

$$\begin{aligned} |f(b) - f(a)| &= |f'(1) - f'(0)| \\ &\leq 8 \int_0^1 \psi^{-1}\left(\frac{4B}{(b-a)^2u^2}\right) p((b-a)du) \\ &= 8 \int_0^{|b-a|} \psi^{-1}\left(\frac{4B}{u^2}\right) p(du) \end{aligned}$$

Let $I(t) = \int_0^1 \psi\left(\frac{|f(t)-f(s)|}{p(t-s)}\right) ds$. There is t_0 such that $I(t_0) \leq B$. With this t_0 we will show that

$$|f(t_0) - f(0)| \leq 4 \int_0^1 \psi^{-1}\left(\frac{4B}{u^2}\right) p(du)$$

and

$$|f(t_0) - f(1)| \leq 4 \int_0^1 \psi^{-1}\left(\frac{4B}{u^2}\right) p(du)$$

so that

$$|f(1) - f(0)| \leq 8 \int_0^1 \psi^{-1}\left(\frac{4B}{u^2}\right) p(du)$$

we will now pick

$$t_0 > u_1 > t_1 > \cdots\cdots > u_n > t_n \cdots$$

recursively as follows. Let $d_n = p(t_{n-1})$ and pick u_n such that $p(u_n) = \frac{d_n}{2}$. Then $\int_0^{u_n} I(t)dt \le B$ and

$$\int_0^{u_n} \psi\left(\frac{|f(t_{n-1} - f(s)|}{p(t_{n-1} - s)}\right) ds \le I(t_{n-1})$$

Now t_n is chosen from $[0, u_n]$ such that $I(t_n) \le \frac{2B}{u_n}$ and

$$\psi\left(\frac{|f(t_{n-1} - f(t_n)|}{p(t_{n-1} - t_n)}\right) \le \frac{2I(t_{n-1})}{u_n}$$

Since $\frac{2I(t_{n-1})}{u_n} \le \frac{4B}{u_{n-1}u_n} \le \frac{4B}{u_n^2}$, we have

$$|f(t_n) - f(t_{n-1})| \le \psi^{-1}\left(\frac{4B}{u_n^2}\right) p(t_{n-1} - t_n) \le \psi^{-1}\left(\frac{4B}{u_n^2}\right) p(t_{n-1})$$

and

$$p(t_{n-1}) = 2p(u_n) = 4\big[p(u_n) - \frac{p(u_n)}{2}\big] \le 4[p(u_n) - p(u_{n+1}]$$

Then

$$|f(t_n) - f(t_{n-1})| \le 4\psi^{-1}\left(\frac{4B}{u_n^2}\right)[p(u_n) - p(u_{n+1})] \le 4\int_{u_{n+1}}^{u_n} \psi^{-1}\left(\frac{4B}{u^2}\right) p(du)$$

Sum over n.

$$|f(t_0) - f(0)| \le 4\int_0^{t_0} \psi^{-1}\left(\frac{4B}{u^2}\right) p(du) \le 4\int_0^1 \psi^{-1}\left(\frac{4B}{u^2}\right) p(du)$$

In a similar fashion, replacing $f(t)$ by $f(1-t)$,

$$|f(t_0) - f(1)| \le 4\int_0^{1-t_0} \psi^{-1}\left(\frac{4B}{u^2}\right) p(du) \le 4\int_0^1 \psi^{-1}\left(\frac{4B}{u^2}\right) p(du)$$

4.3. View in terms of weak convergence. Let μ_N be the measure on $C[0,1]$ induced by the random polygon. If we have a weak limit μ along a sub sequence that will do it.

Uniform tightness. Need K_ϵ compact set in $C[0,1]$ such that

$$\sup_N [\mu_N(K^c)] \le \epsilon$$

Fix N. $0 \le j \le k \le N$. Kolmogorov's inequality.

Let $\{X_i\}$ be mutually independent and have mean 0 and finite variance. Then with $S_n = X_1 + \cdots + X_n$

$$P[\sup_{1\leq j\leq n} |S_j| \geq \ell] \leq \frac{1}{\ell^2} E[S_n^2]$$

Let $B_r = \big\{|S_1, \ldots, |S_{r-1}| < \ell, |S_r| \geq \ell\big\}$

$$\int_{B_r} |S_n|^2 dP = \int_{B_r} [(S_r)^2 + 2S_r(S_n - S_r) + (S_n - S_r)^2] dP \geq \ell^2 P(B_r)$$

You can sum over r and since B_r are disjoint

$$\ell^2 P[\cup_r B_r] \leq \sum_r \int_{B_r} |S_n|^2 dP \leq \int |S_n|^2 dP$$

This is enough to provide the following estimate for a potential approximation

$$P[\sup_{0\leq h\leq\delta} |x(t+h) - x(t)| \geq \epsilon] \leq \frac{C\delta}{\epsilon^2}$$

By Ascoli-Arzela theorem we need to estimate the modulus of continuity. Since $\xi(0) = 0$, uniform boundedness will follow from uniform estimates on the modulus of continuity. Let $\delta > 0$ be given. Divide the interval $[0,1]$ into $\frac{1}{\delta}$ overlapping subintervals of length 2δ. If we control the oscillation in all the subintervals of length 2δ then since any interval of length δ is contained in one of the $\frac{1}{\delta}$ intervals of length 2δ the modulus of continuity at δ is controlled.

But there are $\frac{2}{h}$ such intervals are needed to estimate the modulus of continuity. The estimate misses the mark. Need

$$P[\sup_{s:t\leq s\leq t+\delta} |x(s) - x(t)| \geq \epsilon] \leq \frac{c(\delta)}{\epsilon^2}$$

with $c(\delta) = o(\delta)$. Note that $\xi(t)$ has independent increments and $E[(\xi(t) - \xi(s))^2] = |t-s|$.

Martingales and Doob's inequality.

(Ω, Σ, P) is a probability space, $\Sigma_n \subset \Sigma$. $\uparrow$. A martingale is a sequence $\{X_n\}$ such that X_n is Σ_n measurable, $\in L_1$ and

$$E[X_n|\Sigma_{n-1}] = X_{n-1}$$

or $X_n = Y_1 + \cdots + Y_n$, Y_j is Σ_j measurable and $E[Y_n|\Sigma_{n-1}] = 0$.

If X_n is a martingale with respect to (Ω, Σ_n, P),

$$E[|X_n||\Sigma_{n-1}] \geq |X_{n-1}|$$

It is enough to note that

$$E[\max\{X,Y\}|\Sigma] \geq \max\big\{E[X|\Sigma], E[Y|\Sigma]\big\}$$

Let as before

$$B_r = \{\sup_{j<r} |X_1 + \cdots X_j| < \ell, |X_1 + \cdots X_r| \geq \ell]$$

If $\xi_n = \max\{|X_1|, \ldots, |X_n|\}$, then

$$\ell \int_{\xi_n \geq \ell} dP \leq \sum_r \int_{B_r} |X_r| dP \leq \sum_r \int_{B_r} |X_n| dP = \int_{\xi_n \geq \ell} |X_n| dP$$

Lemma. Let X and Y be two nonnegative random variables such that

$$P[X \geq \ell] \leq \frac{1}{\ell} \int_{X \geq \ell} Y dP$$

Then for $p > 1$,

$$E[X^p] \leq \big(\frac{p}{p-1}\big)^P E[Y^p]$$

Proof. Let $F(t) = P[X \geq t]$ and $G(t) = P[Y \geq t]$

$$\begin{aligned} E[X^p] &= -\int_0^\infty t^p dF(t) \\ &= p\int_0^\infty F(t) t^{p-1} dt \\ &= p\int_0^\infty \int \mathbf{1}_{X \geq t} Y t^{p-2} dP dt \\ &= p\int \int_0^X t^{p-2} Y dt dP \\ &= \frac{p}{p-1} \int X^{p-1} Y dP \\ &\leq \frac{p}{p-1} \|X^{p-1}\|_{p^*} \|Y\|_p \\ &= \frac{p}{p-1} \|X\|_p^{1-\frac{1}{p}} \|Y\|_p \end{aligned}$$

Now we can improve the estimate on martingales

$$P[\sup_{1 \leq j \leq n} |X_j| \geq \ell] \leq \frac{1}{\ell^4} (\frac{4}{3})^4 E[S_n^4]$$

One can ask in general if we have a Markov process with transition probability $p(t, x, dy)$ on some R^d when can we say that the Markov process with transition probability p can be realized in the space of continuous functions.

Let

$$h(t,\epsilon) = \sup_x p(t, xB(x,\epsilon)^c) = \sup_x P[|x(t) - x(0)| \geq \epsilon | x(0) = x]$$

If for every $\epsilon > 0$

$$\lim_{t \to 0} \frac{h(t,\epsilon)}{t} = 0$$

then the process can be realized in $C[[0,T]; R^d]$. We saw before that it is enough to show that

$$h^*(t,\epsilon) = P[\sup_{0 \leq s \leq t} |x(s) - x(0)| \geq \epsilon]$$

satisfies

$$\lim_{t \to 0} \frac{h^*(t,\epsilon)}{t} = 0$$

Lemma.

$$h^*(t, 2\epsilon) \leq \frac{h(t,\epsilon)}{1 - h(t,\epsilon)}$$

We will let the time in $x(\cdot)$ vary over a grid of equally spaced points and get estimates to hold uniformly as the grid size goes to 0.

Let $X_1, \ldots X_n$ be random variables such that for $1 \leq i < j \leq n$

$$\sup_{1 \leq i < j \leq n} P[|X_j - X_i| \geq \epsilon | X_1, \ldots, X_i] \leq \delta$$

a.e. Then

$$P[\sup_{1 \leq i < j \leq n} |X_j - X_i| \geq 4\epsilon] \leq \frac{\delta}{1 - \delta}$$

It is enough to show that

$$P[\sup_{1 \leq i \leq n} |X_i - X_1| \geq 2\epsilon] \leq \frac{\delta}{1 - \delta}$$

Let

$$B = \{\sup_{1 \leq j \leq n} |X_j - X_1| \geq \epsilon\}$$

is the disjoint union over $r = 1, \ldots, n$ of

$$B_r = \Big\{ \sup_{1 \leq j \leq r-1} |X_j - X_1| < 2\epsilon, |X_r - X_1| \geq 2\epsilon \Big\}$$

.

$$B = [B \cap \{|X_n - X_1| < \epsilon\}] \cup [B \cap \{|X_n - X_1| \geq \epsilon\}]$$

$$P[B \cap \{|X_n - X_1| \geq \epsilon\}] \leq P[|X_n - X_1| \geq \epsilon] \leq \delta$$

$$P[B_r \cap \{|X_n - X_1| < \epsilon\}] \leq P[B_r \cap \{|X_n - X_r| \geq \epsilon\}] \leq \delta P[B_r]$$

It follows that

$$P[B] \leq \delta + \delta P[B]$$

and therefore $P(B) \leq \frac{\delta}{1-\delta}$.

The generator $Lf = \lim_{t \to 0} \frac{T_t f - f}{t}$ is local.

Kolmogorov's backward and forward equations.

Assume

$\frac{1}{t}\int_{|y-x|\ge\epsilon} p(t,x,dy) = 0$ locally uniformly in x.

$\frac{1}{t}\int_{|y-x|\le\epsilon} (y-x)p(t,x,dy) = b(x)$ locally uniformly in x.

$\frac{1}{t}\int_{|y-x|\le\epsilon} (y-x)\otimes(y-x)p(t,x,dy) = a(x)$ locally uniformly in x.

$$\lim_{h\to 0}\frac{1}{h}\int [f(y)-f(x)]p(h,x,dy) = (Lf)(x) = \frac{1}{2}\sum a_{i,j}(x)\frac{\partial^2 f}{\partial x_i\partial x_j}(x) + \sum b_j(x)\frac{\partial f}{\partial x_j}(x)$$

locally uniformly in x.

Suppose for $t>0$, $p(t,x,y)$ is smooth in (t,x). Then for fixed y and $t>0$, as a function of t,x

$$\frac{\partial p}{\partial t} = \lim_{h\to 0}\frac{1}{h}[p(t+h,x,y)-p(t,x,y)] = \lim_{h\to 0}\frac{1}{h}\int p(h,x,z)[p(t,z,y)-p(t,x,y)]dy = Lp$$

For fixed x as a function of t and y it will satisfy the forward equation

$$\frac{\partial p}{\partial t} = \frac{1}{2}\sum\frac{\partial^2}{\partial y_i\partial y_j}[a_{i,j}(y)p(t,x,y)] - \sum\frac{\partial}{\partial y_j}[b_j(y)p(t,x,y)]$$

It will be a weak solution.

$$\begin{aligned}\lim_{h\to 0}\int\int[f(x)p(h,x,y)dx - f(y)]g(y)dy \\ &= \lim_{h\to 0}\int\int f(x)g(y)p(h,x,y)dy - \int f(y)g(y)dy \\ &= \lim_{h\to 0}[\int\int f(x)g(y)p(h,x,y)dy - \int f(x)g(x)dx] \\ &= \lim_{h\to 0}\int\int f(x)[g(y)p(h,x,y)-g(x)]dydx \\ &= < f, Lg > = < L^* f, g >\end{aligned}$$

How do you describe a stochastic process. Discrete time. Successive conditionals. Continuos time successive infinitesimal conditionals.

$$\lim_{h\to 0}\int (f(y)-f(x(t)))\mu_{t,\omega,h}(dy) = [L_{t,\omega}f](x(t))$$

$$L_{t,\omega}f = \frac{1}{2}\sum a_{i,j}(t,\omega)\frac{\partial^2 f}{\partial x_i\partial x_j}(x(t)) + \sum b_j(t,\omega)\frac{\partial f}{\partial x_j}(x(t))$$

Infinitely divisible. What is the tangent space to probability distributions on R^d. Infinitely divisible distributions. Levy-Khinchine theorem.

$D(t,\omega)$, Depends as $D(t,x(t))$ Markov. $D(x(t))$. Time homogeneous. D is Gaussian $N(a(t,\omega),b(t,\omega))$ contuous paths. $N(a(t,x(t)),b(t,x(t)))$ Markov. $[a(t),b(t)x(t)+c(t)]$ Gauss-Markov process.

How to rigorously connect the infinitesimal characteristics with the measure?

$$P[X_n \in A|\Sigma_{n-1}] = \nu_n(X_1,\ldots,X_{n-1},A)$$

$$E[f(X_n)-f(X_{n-1})|\Sigma_{n-1}] = \int [f(y)-f(X_{n-1})]\nu_n(X_1,\ldots,X_{n-1},dy)$$

$$E\Big[f(X_n)-f(X_{n-1})-g_n(X_1,\ldots,X_{n-1})|\Sigma_{n-1}\Big] = 0$$

$$Z_n = f(X_n)-f(X_0)-\sum_{j=1}^{n} g_j(X_1,\ldots,X_{j-1})$$

is a martingale.

$$\int \frac{f(y)}{f(X_{n-1})}|\Sigma_{n-1}\nu_n(X_1,\ldots,X_{n-1},dy) = h_n(X_1,\ldots,X_{n-1})$$

$$Z_n = \frac{f(X_n)}{f(X_0)}\prod_{j=1}^{n}\frac{1}{h_j(X_1,\ldots,X_{j-1})}$$

is a martingale.

Continuous versions.

$$f(x(t))-f(x(0))-\int_0^t (L_{s,\omega}f)(x(s))ds$$

is a martingale

$$\frac{f(x(t))}{f(x(0))}\exp\big[-\int_0^t \frac{(L_{s,\omega}f)(x(s))}{f(x(s))}ds\big]$$

is a martingale

One way to model is

$$x(t+h) = x(t)+b(t,\omega)h+Z_h$$

Z_h is a mean 0 Gaussian with dispersion $ha(t,\omega)$, modeled by $\sigma(t,\omega)[B(t+h)-B(t)]$. with $\sigma\sigma^* = a$.

$$dz(t) = b(t,\omega)dt+\sigma(t,\omega)dB(t)$$

In the Markov case

$$dz(t) = b(t,z(t))dt+\sigma(t,z(t))dB(t)$$

Does

$$z(t) = z(0) + \int_0^t b(z(s))ds + \int_0^t \sigma(s, x(s))dB(s)$$

make sense?

What regularity does $B(\cdot)$ have. We saw it was Holder with exponent $\alpha < \frac{1}{2}$. It is NOT Holder $\frac{1}{2}$. Divide $[0,1]$ into n equal parts $\{t_j\}$. $\frac{x(t_j)-x(t_{j-1})}{\sqrt{t_j-t_{j-1}}}$ are independent standard Gaussians and

$$C = \sup_{s,t} |x(t) - x(s)| \sqrt{|t-s} \geq \sup_j \frac{|x(t_j) - x(t_{j-1})|}{\sqrt{|t_j - t_{j-1}}} = \sup[|U_1|, \ldots, |U|_n]$$

When n is large $\sup_j |U_j|$ of n Gaussians is very large with high probability. Therefore $C = \infty$.

$x(t)$ is definitely not diffrentiable. Is it of BV so we can justify the integral? It is not.

$$E[\sum_j [(x(t_j) - x(t_{j-1})]^2] = \sum [t_j - t_{j-1}] = T$$

Variance of

$$\begin{aligned}
[x(t_j) - x(t_{j-1})]^2 &= E[[x(t_j) - x(t_{j-1})]^4] - [E[x(t_j) - x(t_{j-1})]^2]^2 \\
&= 3(t_j - t_{j-1})^2 - (t_j - t_{j-1})^2 \\
&= 2(t_j - t_{j-1})^2
\end{aligned}$$

The sum tends to 0 as the partition is refined. The quadratic variation is T and can not be of BV in any intervals. With some extra work one can show

$$\lim_{n\to\infty} \sup_{0\leq t\leq T} | \sum_{j:\frac{j}{2^n}\leq t} [x(\frac{j+1}{2^n} - x(\frac{j}{2^n})]^2 - t| = 0$$

a.e.